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B Duffield Esquire

George Duffield A.M.

In tali nunquam lassat venatio sylva.
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AN
INTRODUCTION
TO
MENSURATION
AND
PRACTICAL GEOMETRY.

BY

JOHN BONNYCASTLE,

OF THE ROYAL MILITARY ACADEMY, WOOLWICH.

To which is added,

AN APPENDIX,

CONTAINING A CONCISE SYSTEM OF

GAUGING.

THE FIFTH AMERICAN, FROM THE TENTH LONDON EDITION.

WITH CORRECTIONS AND ADDITIONS.

H. DUFFIELD

PHILADELPHIA:
KIMBER & SHARPLESS, NO. 8, SOUTH FOURTH [illegible]
Adam Waldie, Printer.
1829.

Eastern District of Pennsylvania, to wit:

L. S. Be it remembered, That on the twenty-third day of September, in the forty-eighth year of the Independence of the United States of America, A. D. 1823, Kimber & Sharpless, of the said district, have deposited in this office the title of a book, the right whereof they claim as proprietors, in the words following, to wit:

"An Introduction to Mensuration and Practical Geometry. By John Bonnycastle, of the Royal Military Academy, Woolwich. To which is added, an Appendix, containing a concise system of Gauging. Third American, from the tenth London edition. With corrections and additions."

In conformity to the act of the congress of the United States, intituled, "An act for the encouragement of learning, by securing the copies of maps, charts, and books, to the authors and proprietors of such copies, during the times therein mentioned:"—and also to the act, entitled, "An act supplementary to an act, entitled, 'An act for the encouragement of learning, by securing the copies of maps, charts, and books, to the authors and proprietors of such copies during the times therein mentioned,' and extending the benefits thereof to the arts of designing, engraving, and etching historical and other prints."

D. CALDWELL,
Clerk of the Eastern District of Pennsylvania.

Gift
Tappan Presb. Ass.
11-13-1931

PREFACE.

TO THE LONDON EDITION.

THE ART OF MEASURING, like all other useful inventions, appears to have been the offspring of want and necessity; and to have had its origin in those remote ages of antiquity, which are far beyond the reach of credible and authentic history. Egypt, the fruitful mother of almost all the liberal sciences, is imagined likewise to have given birth to GEOMETRY or MENSURATION; it being to the inundations of the Nile that we are said to be indebted for this most perfect and delightful branch of human knowledge.

After the overflowings of the river had deluged the country, and all artificial boundaries and land-marks were destroyed, there could have been no other method of ascertaining individual property, than by a previous knowledge of its figure and dimensions. From this circumstance, it appears highly probable, that Geometry was first known and cultivated by the ancient Egyptians; as being the only science which could administer to their wants, and furnish them with the assistance they required. The name itself signifies properly the *art of measuring the earth;* which serves still further to confirm this opinion, especially as it is well known that many of the ancient mathematicians applied their geometrical knowledge entirely to that purpose, and that even the Elements of Euclid, as they now stand, are only the theory from whence we obtain the rules and precepts of our present more mechanical practice.

But to trace the sciences to their first rude beginnings, is a matter of learned curiosity, which could afford but little gratification to readers in general. It is of much more consequence to the rising generation to be informed that, in their present improved state, they are exceedingly useful and important. And in this respect, the art I have undertaken to elucidate is inferior to none, arithmetic only excepted. Its use in most of the different branches of the Mathematics is so general and extensive, that it may justly be considered as the mother and mistress of all the rest, and the source from whence were derived the various properties and principles to which they owe their existence.

As a testimony of this superior excellence, I need only mention a few of those who have studied and improved it; in which illustrious catalogue we have the names of Euclid, Archimedes, Thales, Anaxagoras, Pythagoras, Plato, Appollonius, Philo, and Ptolemy, amongst the ancients: and Huygens, Wallis, Gregory, Halley, the Bernouillies, Euler, Liebnitz and Newton, amongst the moderns; all of whom applied themselves to particular parts of it, and greatly enlarged and improved the subject. To the latter especially we are indebted for many valuable discoveries in the higher branches of the art; which have not only enhanced its dignity and importance, but rendered the practical application of it more general and extensive.

The degree of estimation in which the art was held by these, and other eminent characters, will, in general, it is apprehended, be thought a sufficient encomium on its merits. But, for the sake of young people, and those of a confined education, it may not be amiss to give a few more instances of its advantage, and show that its importance in trade and business is not inferior to its dignity as a science. Artificers of almost all denominations are indebted to this invention for the establishment of their several occupations, and the perfection and value of their workmanship. Without its assistance all the great and noble works of Art would have been imperfect and useless. By this means the architect lays down his plan, and erects his edifice; bridges are built over large rivers; ships are constructed; and property of all kinds is accurately measured, and justly estimated. In short, most of the elegances and conveniences of life owe their existence to this art, and will be multiplied in proportion as it is well understood, and properly practised.

From this view of the subject, it is hardly tob e accounted for, that, in a commercial nation, like our own, an art of such general application should have been so greatly neglected. Mechanics of all kinds, it is well known, are but ill acquainted with its principles; and those who have been the best qualified to afford them any assistance, have thought it beneath their attention. Till within a few years past there could not be found a regular treatise upon this subject in the English language. Some particular branches, it is true, had been greatly cultivated and improved; but these were only to be found in their miscellaneous state, interspersed through a number of large volumes, in the possession of but a few, and in a form and language totally unintelligible to those for whom they were more immediately necessary.

Dr. Hutton was the first person, in this country, who undertook to collect these scattered fragments, and to treat of the subject in a scientific, methodical manner. A small treatise by Hawney, and some others of little note, had indeed been long in the hands of the public; but these were extremely defective, both in matter and method; neither the principles nor practice of the art being properly or clearly explained. Before the publication of the treatise above mentioned, Mr. Robertson's may be considered as the only book, of any value, that could be consulted, either by the artizan or mathematician; and had he given the theory as well as the practice of the art, and divested his rules and examples of their algebraical form, there would have been no want of any other elementary treatise.

To these two writers I am greatly indebted for many things in the following pages, and am ready to acknowledge, that I have used an unreserved freedom in selecting from their works, wherever I found them to answer my purpose. To Dr. Hutton I am particularly obliged, and am so far from desiring to supersede the use of his performance by this publication. that I only wish it to be thought a useful introduction to it. His treatise is excellent in its kind; and had it been as well calculated for the use of the uninformed Artist as it is for the Mathematician, the following compendium had certainly never been published.

The method I have observed in composing this work, is that which was used in the "*Scholar's Guide to Arithmetic;*" and, as my object has been to facilitate the acquirement of the same kind of useful knowledge, I am not without hopes of its being received with equal candour and approbation.

In school-books, and those designed for the use of learners, it has always appeared to me, that plain and concise rules, with proper exercises, are entirely sufficient for the purpose. In science, as well as in morals, example will ever enforce and illustrate precept; for this reason an operation, wrought out at length, will be found of more service to beginners than all the tedious directions and observations that can possibly be given them. From constant experience I have been confirmed in this idea, and it is in pursuance of it that I have formed the plan of this publication. I have not been ambitious of adding much new matter to the subject: but only to arrange and methodize it in a manner more easy and rational than had been done before.

The text part of the work contains the rules in words at length, with examples to exercise them; and, in order that the learner may not be perplexed and interrupted in his progress, the remarks and demonstrations are confined to the notes, and may be consulted or not, as shall be thought necessary. To those who would wish not to take things upon trust, but to be acquainted with the grounds and *rationale* of the operations they perform, they will be found extremely serviceable: and for this purpose I have endeavoured to make them as easy as the nature of the subject would admit. But they can be consulted only by such as have made a previous acquaintance with several other branches of mathematical learning.

Some of the most difficult rules relating to the surfaces of solids, &c. could not be conveniently given, but by means of algebraical theorems; and as this was foreign to my purpose, I have not scrupled to omit them; being well persuaded that what is done upon that head will be fully sufficient to answer most practical purposes. In the Practical Geometry, likewise, which is prefixed to this treatise, such problems only are introduced as were known to be most intimately connected with the subject. And as this part of the work is a proper and necessary introduction to the rest, I have spared no pains in making it as clear and intelligible as possible.

Upon the whole, I have endeavoured to consult the wants of the learner, more than those of the man of science. And if I have succeeded in this respect, my purpose is answered. I have not sought for reputation as a mathematician, but only to be useful as a tutor.

N. B. The favourable reception this work has met with, has induced me in this edition to make such alterations and additions as have since occurred to me, and which are such as I hope will render it still more acceptable to the public.

Royal Academy, Woolwich,
July 14, 1807.

ADVERTISEMENT

TO THE THIRD AMERICAN EDITION.

THE editor, in preparing the following work for the present edition, has given to the whole a careful revision. A number of errors which existed in the answers to the questions, have been corrected. The answers now given, where decimals occur, are such as are obtained by using a determinative number of them in the calculation. This will afford to the careful student, the satisfaction which results from an agreement of his answer with that in the book.—Improved rules have been given to a few of the problems, and several useful questions have been added. To some of the more intricate questions, solutions have been annexed. The demonstrations which were left out, in the preceding American editions, have been mostly restored. A few problems in solids, which were not thought to be important in an elementary work, have been omitted.

B. HALLOWELL.

West-Town Boarding School,
10th month 1, 1823.

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OF THE

DIFFERENT MEASURES USED IN THIS WORK.

Lineal Measures.	
12 inches make	1 foot.
3 feet.	1 yard.
6 feet.	1 fathom.
16½ feet, or 5½ yards,	1 pole, or rod.
40 poles.	1 furlong.
8 furlongs. . .	1 mile.

Square Measures.	
144 inches make	1 foot.
9 feet.	1 yard.
36 feet.	1 fathom.
272¼ feet. or 30¼ yds.	1 pole. or rod.
1600 poles.	1 furlong.
64 furlongs. . .	1 mile.

Note.—The chain made use of in measuring land, commonly called Gunter's chain, is 4 poles, or 22 yards in length, and consists of 100 equal links, each link being $\frac{22}{100}$ of a yard =.66 of a foot, or 7.92 inches long.

An acre of land is also equal to 10 square chains; that is, 10 chains in length, and 1 in breadth; or it is 4840 square yards, or 160 square poles, or 100,000 square links.

Note also that in Land Measure,	
40 perches, or square poles	make 1 rood.
4 roods.	1 acre.

And in Cubic Measure,	
1728 inches make	1 foot.
27 feet.	1 yard.
166⅜ yards.	1 pole.

Other Measures.

282 cubit inches make.	1 gallon ale measure.
231.	1 gallon wine measure.
268⅘.	1 gallon dry measure.
128 cubic feet, or 8 feet in length and 4 in breadth and 4 in height	1 cord of wood.
34¾ cubic feet, or 16½ feet in length, 1½ in breadth, and 1 in height.	1 perch of stone.

H. DUFFIELD

PRACTICAL GEOMETRY.

DEFINITIONS.

1. GEOMETRY is that science which treats of the descriptions and properties of magnitudes in general.

2. A *point* is that which has position, but not magnitude.

3. A *line* is length without breadth; and its bounds or extremes are points.

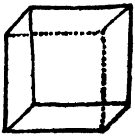
4. A *right line* is that which lies evenly between its extreme points.

---

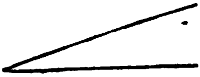
5. A *superficies* is that which has length and breadth only: and its bounds or extremes are lines.

6. A *plane superficies* is that which touches in every part any right line that can be drawn in that superficies.

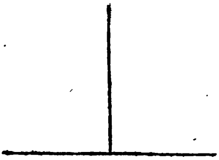
7. A *solid* is that which has length, breadth, and thickness; and its bounds or extremes are superficies.



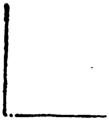
8. A *plane rectilineal angle* is the inclination or opening of two right lines which meet in a point.



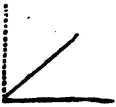
9. One line is said to be *perpendicular* to another, when it makes the angles on both sides of it equal to each other.



10. A *right angle* is that which is formed by two lines that are perpendicular to each other.*



11. An *acute angle* is that which is less than a right angle.



12. An *obtuse angle* is that which is greater than a right angle.

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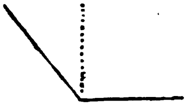
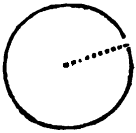
* Any angle differing from a right one, is, by some writers, called an *oblique angle.*

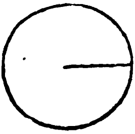


13. A *circle* is a plane figure, formed by the revolution of a right line about one of its extremities, which remains fixed.*

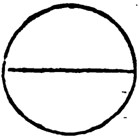


14. The centre of a circle is the point about which it is described; and the circumference is the line or boundary by which it is contained.

15. The *radius* of a circle is a right line drawn from the centre to the circumference.



16. The *diameter* of a circle is a right line passing through the centre, and terminated both ways by the circumference.



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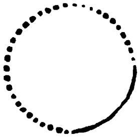
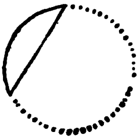
* N. B. The circumference itself, for the sake of conciseness, is sometimes called a circle.

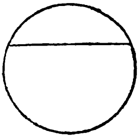
17. An *arc* of a circle is any part of its periphery or circumference.



18. A *chord* is a right line joining the extremities of an arc.

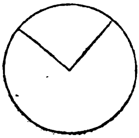


19. A *segment* of a circle is any part of a circle bounded by an arc and its chord.

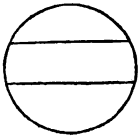


20. A *sector* is any part of a circle bounded by an arc and its two radii drawn to its extremities.

N. B. A *semicircle* is half a circle, and a *quadrant* the quarter of it.



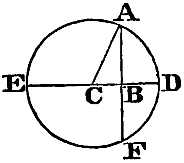
21. A *zone* is a part of a circle included between two parallel chords and their intercepted arcs.



22. A *sine* of an arc is a right line drawn from one extremity of an arc perpendicular to a diameter passing through the other extremity, as AB.

23. The *versed sine* of an arc is that part of the diameter which is intercepted between the sine and the arc; as BD or BE.

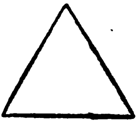
24. The *cosine* of an arc, is that part of the diameter intercepted between the sine and centre, as CB, and is always equal to the difference between the versed sine and the radius.



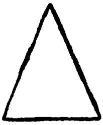
Note. The height of a segment is the part of a diameter contained between the middle of the chord and the arc: and the difference between this and the radius is sometimes called the central distance.

25. All plane figures bounded by three right lines are called *triangles*.

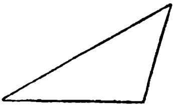
26. An *equilateral triangle* is that whose three sides are all equal.



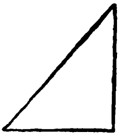
27. An *isosceles triangle* is that which has only two of its sides equal.



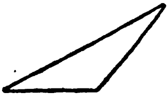
28. A *scalene triangle* is that which has all its three sides unequal.



29. A *right-angled triangle* is that which has one right angle.*



30. An *obtuse-angled triangle* is that which has one obtuse angle.



31. An *acute-angled triangle* is that which has all its angles acute.

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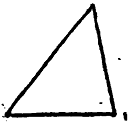
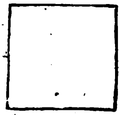
* Any triangle differing from a right-angled one, is called *an oblique-angled triangle.*

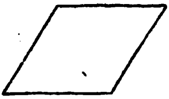


32. All plane figures, bounded by four right lines, are called *quadrangles* or *quadrilaterals*.

33. A *square* is a quadrilateral, whose sides are all equal, and its angles all right angles.



34. A *rhombus* is a quadrilateral, whose sides are all equal, but its angles not right angles.*



35. *Parallel right lines* are such as are in the same plane, and which being produced ever so far both ways, do not meet.

36. A *parallelogram* is a quadrilateral whose opposite sides are parallel.

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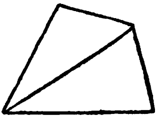
* This figure, by working mechanics, is sometimes called a lozenge.

37. A *rectangle* is a parallelogram whose angles are all right angles.

38. A *rhomboid* is a parallelogram whose angles are not right angles.

39. All other four-sided figures, besides these, are called *trapeziums*.

40. A right line joining any two opposite angles of a four-sided figure is called the *diagonal*.



41. All plane figures contained under more than four sides are called *polygons*.

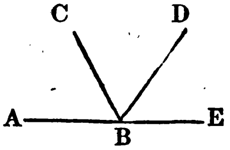
42. Polygons having five sides, are called *pentagons;* those of six sides, *hexagons*; those of seven, *heptagons;* and so on.

43. A *regular polygon* is that whose angles as well as sides are all equal.

44. The *base* of any figure is that side on which it is supposed to stand; and the *altitude* is the perpendicular falling upon it from the opposite angle.

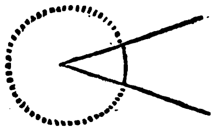
45. In a right-angled triangle the side opposite to the right angle is called the *hypothenuse;* and the other two sides are called *legs*.

46. An angle is usually denoted by three letters, the one which stands at the angular point being always to be read in the middle.



*47. The circumference of every circle is supposed to be divided into 360 equal parts, called *degrees;* each degree into 60 equal parts, called *minutes;* and so on.

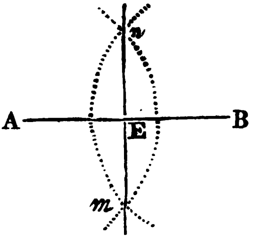
48. The *measure* of any right-lined angle is an arc of a circle contained between the two lines which form that angle, the angular point being the centre.



Note. The angle is estimated by the number of degrees contained in the arc; whence a right angle is an angle of 90 degrees, or ¼ of the circumference.

PROBLEM I.†

To divide a given line AB *into two equal parts.*



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* This and the following definition are used only in Practical Geometry.

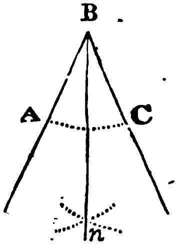
† The demonstrations of most of these problems may be found in Euclid's Elements.

1. From the points A and B, as centres, with any distance greater than half AB, describe arcs cutting each other in *n* and *m*.

2. Through these points, draw the line *n*E*m*, and the point E, where it cuts AB, will be the middle of the line required.

PROBLEM II.

To divide a given angle ABC *into two equal parts.*



1. From the point B, with any radius, describe the arc AC.

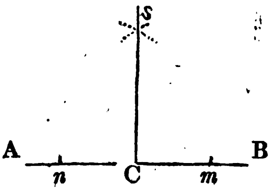
2. And from AC, with the same, or any other radius, describe arcs cutting each other in *n*.

3. Through the point *n* draw the line B*n*, and it will bisect the angle ABC, as was required.

PROBLEM III.

From a given point C, *in a given right line* AB, *to erect a perpendicular.*

Case I. *When the point is near the middle of the line.*

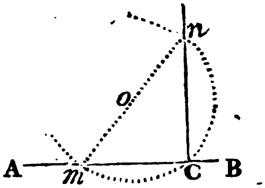


1. On each side of the point C take any two equal distances C*n*, C*m*.

2. From *n* and *m*, with any radius greater than C*n* or C*m*, describe arcs cutting each other in *s*.

3. Through the point *s*, drawn the line *s*C, and it will be the perpendicular required.

CASE II. *When the point is at, or near, the end of the line.*

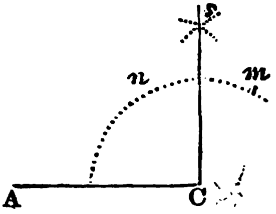


1. Take any point *o*, and with the radius or distance *o*C, describe the arc *m*C*n*, cutting AB in *m* and C.

2. Through the centre *o*, and the point *m*, draw the line *mon*, cutting the arc *m*C*n* in *n*.

3. From the point *n*, draw the line *n*C, and it will be the perpendicular required.

Another method.



1. From the point C, with any radius, describe the arc *rnm*, cutting the line AC in *r*.

2. With the same radius, and *r* as a centre, cross the arc in *n*; and from *n* in like manner, cross it in *m*.

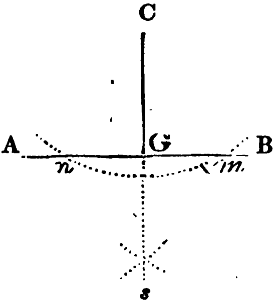
3. From the points *n* and *m*, with the same, or any other radius, describe arcs cutting each other in *s*.

4. Through the point *s*, draw the line *s*C, and it will be the perpendicular required.*

PROBLEM IV.

From a given point C, *out of a given line* AB, *to let fall a perpendicular.*

Case I. *When the point is nearly opposite to the middle of the line.*



1. From the point C, with any radius, describe the arc *nm* cutting AB in *n* and *m*.

2. From the points *n*, *m*, with the same or any other radius, describe two arcs cutting each other in *s*.

3. Through the points C, *s*, draw the line CG*s*, and CG will be the perpendicular required.

Case II. *When the point is nearly opposite to the end of the line.*

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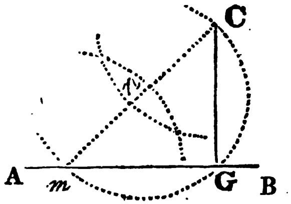
* For another method of raising a perpendicular from any point in a given line, see Prob. XXXIX.



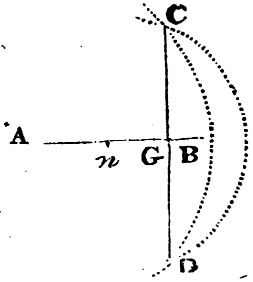
1. To any point *m*, in the line AB, draw the line C*m*.

2. Bisect the line C*m*, or divide it into two equal parts, in the point *n*.

3. From *n*, with the radius *nm*, or *n*C, describe the arc CG*m*, cutting AB in G.

4. Through the point C, draw the line CG, and it will be the perpendicular required.

Another method.



1. From A, or any other point in AB, with the radius AC, describe the arc CD.

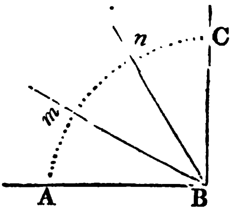
2. And from any other point *n*, in AB, with the radius *n*C, describe another arc cutting the former in C, D.

3. Through the points C, D, drawn the line CGD, and CG will be the perpendicular required.

N. B. Perpendiculars may be more easily raised, and let fall, in practice, by means of a square, or other proper instrument.

PROBLEM V.

To trisect, or divide a right angle ABC *into three equal parts.*



1. From the point B, with any radius BA, describe the arc AC, cutting the legs BA, BC, in A, C.

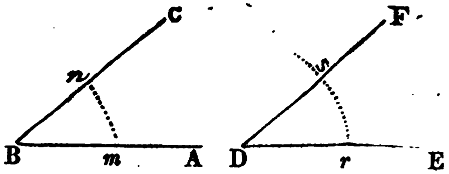
2. And from the point A, with the radius AB, or BC, cross the arc AC in *n*.

3. Also with the same radius, from the point C, cross it in *m*.

4. Through the points *m*, *n*, draw the lines B*m*, B*n*, and they will trisect the angle as was required.

PROBLEM VI.

At a given point D *to make an angle equal to a given angle* ABC.



1. From the point B, with any radius, describe the arc *nm*, cutting the legs BA, BC, in the points *m*, *n*.

2. Draw the line DE, and from the point D, with the same radius as before, describe the arc *rs*.

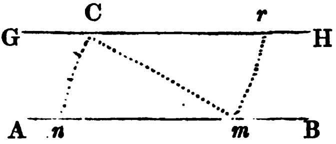
3. Take the distance *mn*, on the former arc, and apply it to the arc *rs*, from *r* to *s*.

4. Through the points D, *s*, draw the line DF, and the angle EDF will be equal to the angle ABC, as was required.

PROBLEM VII.

To draw a line parallel to a given line AB.

Case I. *When the parallel line is to pass through a given point* C,



1. To AB, from the point C, draw any right line C*m*.

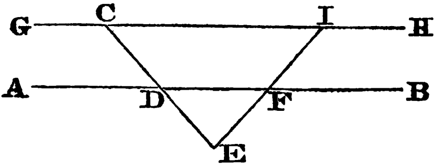
2. From the point *m*, with the radius *m*C, describe the arc C*n*, cutting AB in *n*.

3. And with the same radius, from the point C, describe the arc *mr*.

4. Take the distance C*n*, and apply it to the arc *mr*, from *m* to *r*.

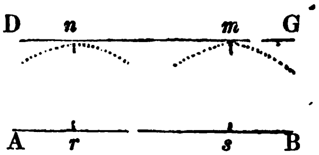
5. Through the points C, *r*, draw the line GC*r*H, and it will be parallel to AB, as was required.

Another method.



1. From C, draw any line CDE, and make DE equal to DC.
2. From E, draw any line EFI, and make FI equal to EF.
3. Through C, I, draw the line GCIH, and it will be parallel to AB.

CASE II. *When the parallel line is to be at a given distance from* AB.



1. From any two points *r*, *s*, in the line AB, with a radius equal to the given distance, describe the arcs, *n*, *m*.
2. Draw the line DG, to touch those arcs without cutting them, and it will be parallel to AB, as was required.

N. B. The former case of this problem, as well as several other operations in Practical Geometry, may be more easily effected by means of the parallel ruler.*

PROBLEM VIII.

To divide a given line AB *into any proposed number of equal parts.*

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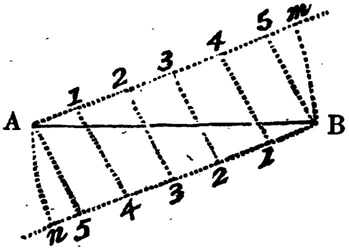
* This ruler may be had of all sizes, but is usually put into a portable case, with a drawing-pen, scale, compasses, and other useful instruments.



1. From one end of the line A, draw A*m*, making any angle with AB; and from B, the other end, draw B*n*, making an equal angle AB*n*.

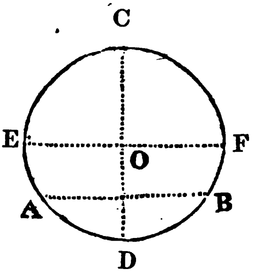
2. In each of the lines A*m*, B*n*, beginning at A and B, set off as many equal parts, of any length, as AB is to be divided into.

3. Join the points A5, 1 4, 2 3, &c. and AB will be divided as was required.

Note. B*n* may be drawn parallel to A*m*, by means of a parallel ruler.

PROBLEM IX.

*To find the centre of a given circle, or one already described.**



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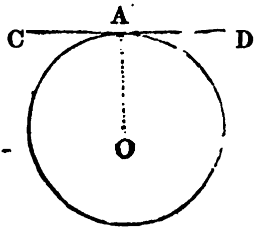
* The centre of a given circle, or of any arc of it, may also be found by joining three points in the circumference, and proceeding as in Prob. XXIV.

1. Draw any chord AB, and bisect it with the perpendicular CD.
2. Bisect CD, in like manner, with the chord EF, and their intersection O, will be the centre required.

PROBLEM X.

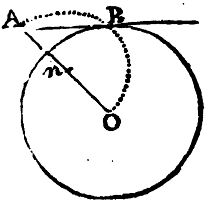
To draw a tangent to a given circle, that shall pass through a given point A.

Case I. *When the point* A *is in the circumference of the circle.*



1. From the given point A, to the centre of the circle, draw the radius AO.
2. Through the point A, draw CD perpendicular to OA, and it will be the tangent required.

Case II. *When the point* A *is without the circle.*



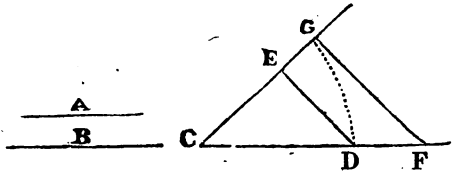
1. To the point A, from the centre O, draw the line OA, and bisect it in *n*.

2. From the point *n*, with the radius *n*A, or *n*O, describe the semicircle ABO, cutting the given circle in B.

3. Through the points A, B, draw the line AB, and it will be the tangent required.

PROBLEM XI.

To two given lines, A, B, *to find a third proportional.*



1. From the point C draw two right lines, making any angle FCG.

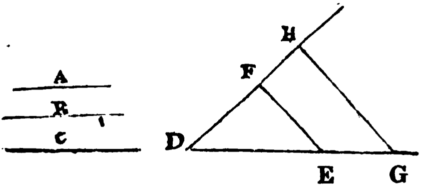
2. In these lines take CE equal to the first term A, and CG, CD, each equal to the second term B.

3. Join ED, and draw GF parallel to it; and CF will be the third proportional required.

That is CE(A): CG (B) :: CD(B): CF.

PROBLEM XII.

To three given right lines, A, B, C, *to find a fourth proportional.*



1, From the point D draw two right lines, making any angle GDH.

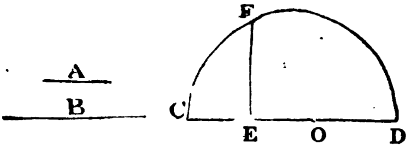
2. In these lines take DF equal to the first term A, DE equal to the second B, and DH equal to the third C.

3. Join FE, and drew HG parallel to it, and DG will be the fourth proportional required.

That is DF(A) : DE(B) :: DH(C) : DG.

PROBLEM XIII.

Between two given right lines A, B, *to find a mean proportional.*



1. Draw any right line, in which take CE equal to A, and ED equal to B.

2. Bisect CD in O, and with OD or OC, as radius, describe the semicircle CFD.

3. From the point E draw EF perpendicular to CD, and it will be the mean proportional required.

That is CE(A) : EF : : EF : ED(B).

PROBLEM XIV.

To divide a given line A B *in the same proportion that another given line* C *is divided.**

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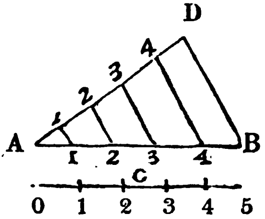
* The case of this problem which most frequently occurs, is that in which the given line is required to be divided into two parts that shall have a given ratio; which may be done in nearly the same manner as above.



1. From the point A draw AD equal to the given line C, and making any angle with AB.

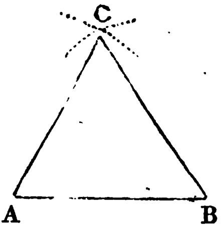
2. To AD apply the several divisions of C, and join DB.

3. Draw the lines 4 4, 3 3, &c. each parallel to DB, and the line AB will be divided as was required.

That is, the parts A1, 1 2, 2 3, 3 4, 4B, on the line AB, will be proportional to the parts 0 1, 1 2, 2 3, 3 4, 4 5, on the line C.

PROBLEM XV.

Upon a given right line AB, *to make an equilateral triangle.*



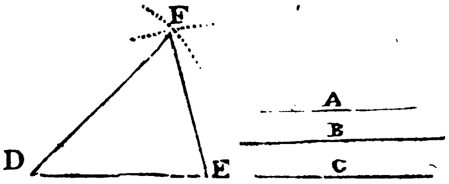
1. From the points A and B, with a radius equal to AB describe arcs cutting in C.

2. Draw the lines AC, BC, and the figure ACB will be the triangle required.

Note. An isosceles triangle may be formed in the same manner, by taking any distance for radius.

PROBLEM XVI.

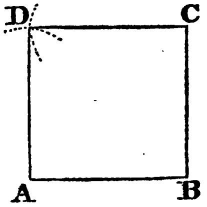
To make a triangle whose three sides shall be respectively equal to three given lines, A, B, C.*



1. Draw a line DE equal to one of the given lines C.
2. On the point D, with a radius equal to B, describe an arc.
3. And on the point E, with a radius equal to A, describe another arc, cutting the former in F.
4. Draw the lines DF, EF, and DFE will be the triangle required.

PROBLEM XVII.

Upon a given line AB *to describe a square.*



1. From the point B, draw BC perpendicular, and equal to AB.

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* The three given lines must be of such lengths that any two of them taken together shall be greater than the third.

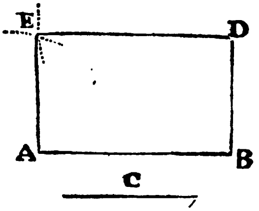
2. On A and C, with the radius AB, describe two arcs cutting each other in D.

3. Draw the lines AD, CD, and the figure ABCD will be the square required.

Note. A rhombus may be made on the given line AB in exactly the same manner, if BC be drawn with the proper obliquity, instead of perpendicularly.

PROBLEM XVIII.

To describe a rectangle, whose length and breadth shall be equal to two given lines AB *and* C.



1. At the point B, in the given line AB, erect the perpendicular BD, and make it equal to C.

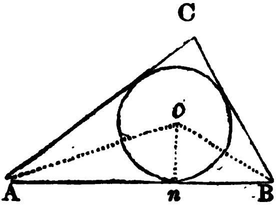
2. From the points D, A, with the radii AB and C describe two arcs cutting each other in E.

3. Join EA and ED, and ABDE will be the rectangle required.

Note. A parallelogram may be described in nearly the same manner.

PROBLEM XIX.

In a given triangle ABC *to inscribe a circle.*

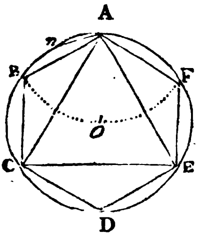


1. Bisect the angles A and B with the lines AO and BO.

2. From the point of intersection O let fall the perpendicular O*n*, and it will be the radius of the circle required.

PROBLEM XX.

In a given circle to inscribe an equilateral triangle, a hexagon, or a dodecagon.



For the hexagon.

1. From any point A as a centre, with a distance equal to the radius AO, describe the arc BOF.

2. Join the points AB, or AF, and either of these lines being carried six times round the circle will form the hexagon required.

That is, the radius of the circle is equal to the side of the hexagon; and the sides of the hexagon divide the circumference of the circle into six equal parts, each containing 60 degrees.

For the equilateral triangle.

1. From the point A, to the second and fourth divisions, or angles of the hexagon, draw the lines AC, AE.

2. Join the points CE, and ACE will be the equilateral triangle required; the arc AC being one third of the circumference, or 120 degrees.

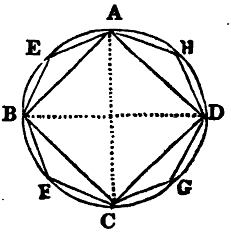
For the dodecagon.

Bisect the arc AB of the hexagon in the point *n*, and the line A*n* being carried twelve times round the circumference, will form the dodecagon required, the arc A*n* being 30 degrees.

If A*n* be again bisected, a polygon may be formed of 24 sides; and by another bisection a polygon of 48 sides; and so on.

PROBLEM XXI.

To describe a square, or an octagon, in a given circle.



For the square.

1. Draw the diameters BD and AC, intersecting each other at right angles.
2. Join the points AB, BC, CD, and DA, and ABCD will be the square required.

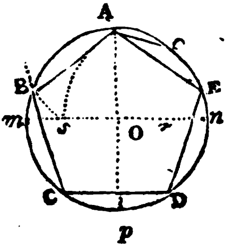
For the octagon.

Bisect the arc AB of the square in the point E, and the line AE being carried eight times round the circumference will form the octagon.

If the arc AE be again bisected, a polygon may be formed of 16 sides: and by another bisection, a polygon of 32 sides; and so on.

PROBLEM XXII.*

To inscribe a pentagon, or decagon, in a given circle.



For the pentagon.

1. Draw the diameters A*p*, *nm*, at right angles to each other, and bisect the radius O*n* in *r*.

2. From the point *r*, with the distance *r*A, describe the arc A*s*, and from the point A, with the distance A*s*, describe the arc *s*B.

3. Join the points A, B, and the line AB being carried five times round the circle, will form the pentagon required.

For the decagon.

Bisect the arc AE of the pentagon in *c*, and the line A*c* being carried ten times round the circumference will form the decagon required.

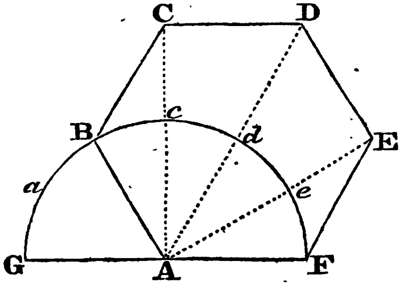
If the arc AC be again bisected, a polygon of 20 sides may be formed; and by another bisection a polygon of 40 sides; and so on.

---

* Besides the figures here constructed, and those arising from thence by continual bisections, or taking the differences, no other regular polygon can be described, from any known method *purely geometrical.*

PROBLEM XXIII.

On a given line AF, *to describe a regular polygon of any proposed number of sides.*



1. From the point A, with the distance AF, describe the semicircle FBG, which divide into as many equal parts G*a*, *a*B, B*c*, &c. as the polygon is to have sides.*

2. From A to the second point of division draw AB, and through the other points *c, d, e,* &c. draw the lines AC, AD, AE, &c.

3. Apply the distance AF from F to E, from E to D, from D to C, &c. and join BC, CD, DE, &c. and ABCD, &c. will be the regular polygon required.

PROBLEM XXIV.

About a given triangle ABC *to circumscribe a circle.*

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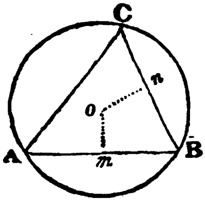
* The semicircle is conveniently divided by means of a scale of chords, with the use of which the student is supposed to be acquainted.



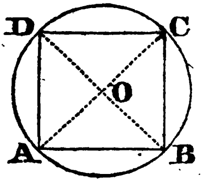
1. Bisect the two sides AB, BC, with the perpendiculars *mo*, and *no*.

2. From the point of intersection *o*, with the distance OA, OB or OC, describe the circle ACB, and it will be that required.

If any two of the angles be bisected, instead of the sides, the intersection of the lines will also give the centre of the inscribed circle.

PROBLEM XXV.

About a given square ABCD *to circumscribe a circle.*

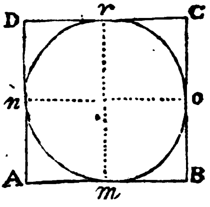


1. Draw the two diagonals AC and BD intersecting each other in O.

2. From the point O, with the distance OA, OB, OC or OD, describe the circle ABCD, and it will be that required.

PROBLEM XXVI.

To circumscribe a square about a given circle.

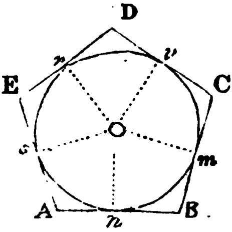


1. Draw any two diameters *no* and *rm* at right angles to each other.

2. Through the points *m*, *o*, *r*, *n*, draw the lines AB, BC, CD, and DA, perpendicular to *rm* and *no*, and ABCD will be the square required.*

PROBLEM XXVII.

About a given circle to circumscribe a pentagon.



1. Inscribe a pentagon in the circle; or, which is the same thing, find the points *m*, *n*, *v*, *r*, *s*, as in Prob. XXII.

2. From the centre *o* to each of these points, draw the radii *on*, *om*, *ov*, *or*, and *os*.

3. Through the points *n*, *m*, draw the lines AB, BC, perpendicular to *on*, *om*; producing them till they meet each other at B.

4. In the same manner, draw the lines CD, DE, EA, and ABCDE will be the pentagon required.

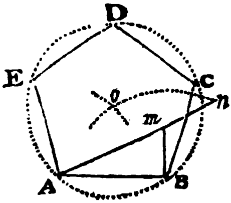
---

* If each of the quadrants *rn*, *mn*, *mo* and *or*, be bisected, and tangents be drawn to those points, the circumscribing figure will be an octagon.

Note.—Any other polygon may be made to circumscribe a circle, by first inscribing a similar one, and then drawing tangents to the circle at the angular points.

PROBLEM XXVIII.*

On a given line AB *to make a regular pentagon.*



1. Make B*m* perpendicular to AB, and equal to one half of it.

2. Draw A*m*, and produce it till the part *mn* is equal to B*m*.

3. From A and B as centres, with the radius B*n*, describe arcs cutting each other in *o*.

4. And from the point *o*, with the same radius or with *o*A, or *o*B, describe the circle ABCDE.

5. Apply the line AB five times round the circumference of this circle, and it will form the pentagon required.

Note.—If tangents be drawn through the angular points A, B, C, D, E, a pentagon circumscribing the circle will be formed; and if the arcs be bisected, a circumscribing decagon may be formed.

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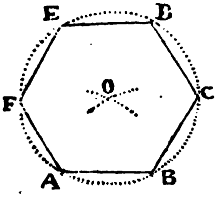
* In the former edition of this work, another method of describing a pentagon was given, as first proposed by Albertus Durer, in his Geometry, p. 55, printed in 1532; but as that is only an approximation, and is not more easy in practice than the present one, which is perfectly accurate, it is here omitted.

PROBLEM XXIX.

On a given line AB *to make a regular hexagon.*



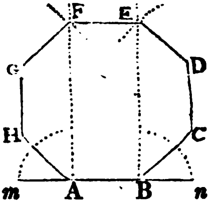
1. From the points A, B, as centres, with the radius AB, describe arcs cutting each other in O.

2. And from the point O, with the distance OA or OB, describe the circle ABCDEF.

3. Apply the line AB six times round the circumference, and it will form the hexagon required.*

PROBLEM XXX.

On a given line AB *to form a regular octagon.*



1. On the extremes of the given line AB erect the indefinite perpendiculars AF and BE.

2. Produce AB both ways to *m* and *n*, and bisect the angles *m*AF and *n*BE with the lines AH and BC.

---

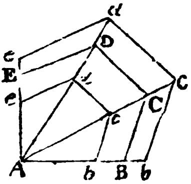
* This construction is founded on the principle, that the radius of every circle is equal to the side of the inscribed hexagon, or the chord of 60°.

3. Make AH and BC equal to AB, and draw HG, CD, parallel to AF or BE, and also each equal to AB.

4. From G, D, as centres, with a radius equal to AB, describe arcs crossing AF, BE, in F, and E; and if GF, FE, and ED be drawn, ABCDEFGH will be the octagon required.

PROBLEM XXXI.

To make a figure similar to a given figure, ABCDE.



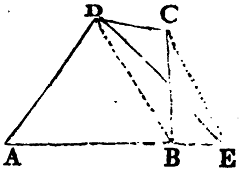
1. Take A*b* equal to the side of the figure required, and from the angle A draw the diagonals A*c* A*d*.

2. From the points *b*, *c*, *d*, draw *bc*, *cd*, *de*, parallel to BC, CD, DE, and A*bcde* will be similar to ABCDE.

The same thing may also be done by making the angles *b*, *c*, *d*, *e*, respectively, equal to the angles B, C, D, E.

PROBLEM XXXII.

To make a triangle equal to a given trapezium ABCD.

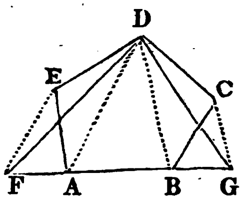


1. Draw the diagonal DB, and make CE parallel to it, meeting the side AB produced in E.

2. Join the points D, E, and ADE will be the triangle required.

PROBLEM XXXIII.

To make a triangle equal to any right lined figure, ABCDEA.



1. Produce the side AB both ways at pleasure.

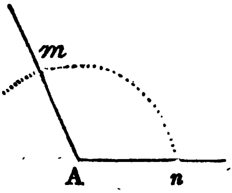
2. Draw the diagonals DA, DB, and parallel to them the lines EF and CG.

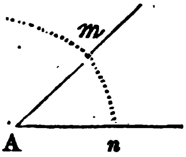
3. Join the points DF, DG, and DFG will be the triangle required.

And in nearly the same manner may any right lined figure whatever be reduced to a triangle.

PROBLEM XXXIV.*

To make an angle of any proposed number of degrees.





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* The line of chords made use of in the following problems, is commonly put upon the plane scale, and is adapted to 90 degrees or the fourth part of a circle.

For a description of this and other instruments made use of in Practical Geometry, see *Mr. Robertson's Treatise on such mathematical instruments as are usually put into a portable case.*

1. Take the first 60 degrees from the scale of chords, and from the point A, with this radius, describe the arc *nm*.

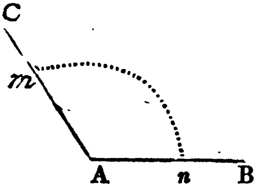
2. Take the chord of the proposed number of degrees from the same scale, and apply it from *n* to *m*.

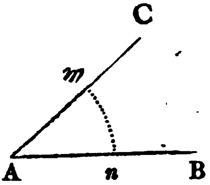
3. From the point A draw the lines A*n* and A*m*, and they will form the angle required.

Note.—Angles greater than 90° are usually taken at twice.

PROBLEM XXXV.*

An angle BAC *being given, to find the number of degrees it contains.*





1. From the angular point A, with the chord of 60 degrees, describe the arc *nm*, cutting the legs in the points *n* and *m*.

2. Take the distance *nm*, and apply it to the scale of chords, and it will show the degrees required.

Note.—When the distance *nm* is greater than 90°, it must be taken at twice, as before.

PROBLEM XXXVI.

In a given circle to inscribe a polygon of any proposed number of sides.

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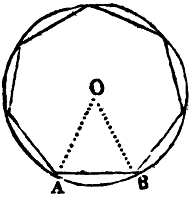
* Both this and the last problem may be performed by means of a *protractor*.

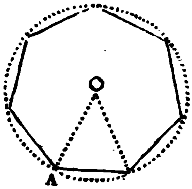


1. Divide 360° by the number of sides, and make an angle AOB, at the centre, whose measure shall be equal to the degrees in the quotient.

2. Join the points AB, and apply the chord AB to the circumference the given number of times, and it will form the polygon required.

PROBLEM XXXVII.

On a given line AB *to form a regular polygon of any proposed number of sides.*



1. Divide 360° by the number of sides, and subtract the quotient from 180 degrees.

2. Make the angles ABO and BAO each equal to half the difference last found.

3. From the point of intersection O, with the distance OA or OB, describe a circle.

4. Apply the chord AB to the circumference the proposed number of times, and it will form the polygon required.*

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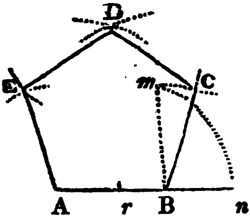
* By this method the circumference of a circle may also be divided into any number of equal parts; for if 360 be divided by the

PROBLEM XXXVIII.*

Upon a right line AB *to describe a regular pentagon.*



1. Produce AB towards *n*, and at the point B make the perpendicular B*m* equal to AB.

2. Bisect AB in *r*, and from *r* as a centre, with the radius *rm*, describe the arc *mn*, cutting AB in *n*.

3. From the points A and B, with the radius A*n*, describe arcs cutting each other in D.

4. And from the points A, D, and B, D, with the radius AB, describe arcs cutting each other in C and E.

5. Join BC, DC, DE, and EA, and ABCDE will be the pentagon required.

This method differs but little from that of Problem XXVIII, and is equally easy and convenient in practice.

PROBLEM XXXIX.

To raise a perpendicular from any point B *in a given line* AB.

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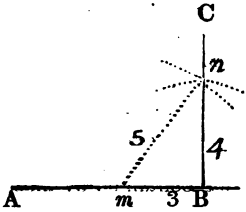
number of parts, and the angle AOB be made equal to the degrees in the quotient, the arc AB will be one of the equal parts required.

* This and the following problem were not given in the first edition of this work, but are now added on account of their elegance and utility. The second is derived from the 47th Prop. B. I. Euclid's Elements, and the first is proposed for a demonstration in the Ladies' Diary for the year 1786.



1. From any scale of equal parts take a distance equal to 3 divisions, and set it from B to *m*.

2. And from the points B and *m*, with the distances 4 and 5, taken from the same scale, describe arcs cutting each other in *n*.

3. Through the points *n*, B, draw the line BC, and it will be the perpendicular required.

Explanation of the characters made use of in the following part of this work.

+ Is the sign of addition.
— ———— of subtraction.
× ———— of multiplication.
÷ ———— of division.
√ ———— of the square root.
∛ ———— of the cube root.
= ———— of equality.
: :: : ———— of proportion.

REMARKS.

1. An angle in a semicircle is a right angle.

2. All angles in the same segment of a circle are equal.

3. Triangles that have all the three angles of the one respectively equal to the three angles of the other, are called equiangular triangles, or similar triangles.

4. In similar triangles the like sides, or sides opposite to the equal angles, are proportional.

5. The areas or spaces of similar triangles are to each other as the squares of their like sides.

6. The areas of circles are to each other as the squares of their diameters, radii, or circumferences.

7. Similar figures are such as have the same number of sides, and the angles contained by those sides respectively equal.

8. The areas of similar figures are to each other as the squares of their like sides.

OF THE

MENSURATION

OF

SUPERFICIES.

THE *area* of any figure is the measure of its surface, or the space contained within the bounds of that surface, without any regard to thickness.

A square whose side is one inch, one foot, or one yard, &c. is called the *measuring unit*, and the area or content of any figure is computed by the number of those squares contained in that figure.

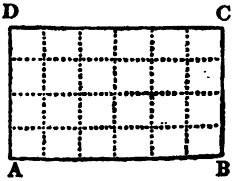
N. B. In all questions involving decimals, they are carried out to the fourth place inclusive, and then taken to the nearest figure; that is, if it be found that, by extending the operation, the next figure would be 5, or upwards, the fourth decimal figure is increased by 1. The student by observing this rule will generally find his results to agree with those given in the book.

PROBLEM I.

To find the area of a parallelogram; whether it be a square, a rectangle, a rhombus, or a rhomboides.

RULE.*

Multiply the length by the perpendicular height, and the product will be the area.

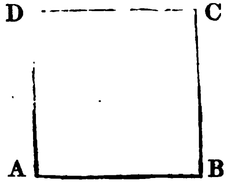


* Take any rectangle ABCD, and divide each of its sides respectively, into as many equal parts as are expressed by the number of times they contain the linear measuring unit, and let all the opposite points of division be connected by right lines. Then, it is evident, that these lines divide the rectangle into a

Note.—The perpendicular height of the parallelogram is equal to the area divided by the base.

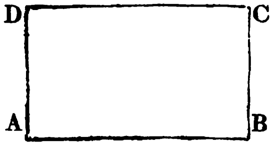
EXAMPLES.

1. Required the area of the square ABCD whose side is 5 feet 9 inches.



Here 5 *fe.* 9 *in.*=5.75: *and* $\overline{5.75}|^2$ =5.75×5.75=33.0625 *feet*=33 *fe.* 0 *in.* 9 *pa.*=*area required.*

2. Required the area of the rectangle ABCD, whose length AB is 13.75 chains, and breadth BC 9.5 chains.



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number of squares each equal to the superficial measuring unit, and that the number of these squares, or the area of the figure, is equal to the number of linear measuring units in the length, as often repeated as there are linear measuring units in the breadth or height, that is, equal to the length multiplied by the height, *which is the rule.*

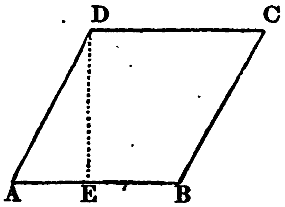
And since a rectangle is equal to an oblique parrallelogram standing upon the same base, and between the same parallels, (Euc. I 35,) the rule is true for any parallelogram in general. Q. E. D.

Rule II. If any two sides of a parallelogram be multiplied together, and the product again by the natural sine of the included angle, the last product will give the area of the parallelogram. That is AB × BC=nat. sine of the angle B=area.

Note.—Because the angles of as quare and rectangle are each 90°, whose natural sine is unity, or 1, the rule in this case is the same as that given in the text.

Here $13.75 \times 9.5 = 130.625$; *and* $\frac{130.625}{10} = 13.0625$ *ac.* = 13 *ac.* 0 *ro.* 10 *po.* = *area required.*

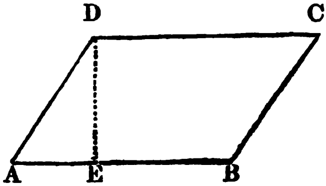
3. Required the area of the rhombus ABCD, whose length AB is 12 feet 6 inches, and its height DE 9 feet 3 inches.



Here 12 *fe.* 6 *in.* = 12.5, *and* 9 *fe.* 3 *in.* = 9.25.

Whence $12.5 \times 9.25 = 115.625$ *fe.* = 115 *fe.* 7 *in.* 6 *pa.* = *area required.*

4. What is the area of the rhomboides ABCD whose length AB is 10.52 chains, and height DE 7.63 chains?



Here $10.52 \times 7.63 = 80.2676$; *and* $\frac{80.2676}{10} = 8.02676$ *acres* = 8 *ac.* 0 *ro.* 4 *po.* = *area required.*

5. What is the area of a square whose side is 35.25 chains?
ac. *ro.* *po.*
Ans. 124 1 1

6. What is the area of a square whose side is 8 feet 4 inches?
fe. *in.* *pa.*
Ans. 69 5 4

7. What is the area of a rectangle whose length is 14 feet 6 inches, and breadth 4 feet 9 inches? *fe. in. pa.*
Ans. 68 10 6

8. Required the area of a rhombus, the length of whose side is 12.24 feet, and height 9.16 feet. *fe. in. pa.*
Ans. 112 1 5

9. Required the area of a rhomboides whose length is 10.51 chains, and breadth 4.28 chains. *ac. ro. pa.*
Ans. 4 1 39.7

10. What is the area of a rhomboides whose length is 7 feet 9 inches, and height 3 feet 6 inches? *fe. in. pa.*
Ans. 27 1 6

11. To find the area of a rectangular board, whose length is 12½ feet, and breadth 9 inches. Ans. 9⅜ feet.

PROBLEM II.

To find the area of a triangle.

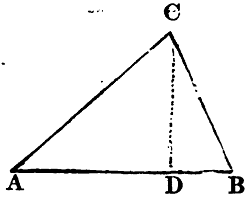
RULE.*

Multiply the base by the perpendicular height, and half the product will be the area.

Note.—The perpendicular height of the triangle is equal to twice the area divided by the base.

EXAMPLES.

1. Required the area of the triangle ABC, whose base AB is 10 feet 9 inches, and height DC 7 feet 3 inches.



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* A triangle is half a parallelogram of the same base and altitude, (Euc. I. 41,) and therefore the truth of this rule is evident.

Here 10 *fe.* 9 *in.*=10.75, *and* 7 *fe.* 3 *in.*=7.25;

Whence 10.75 × 7.25=77.9375, *and* $\frac{77.9375}{2}$=38.96875 *feet*=38 *fe.* 11 *in.* 7½ *pa.*= *area required.*

2. What is the area of a triangle whose base is 18 feet 4 inches, and height 11 feet 10 inches? *fe.* *in.* *pa.*
Ans. 108 5 8

3. What is the area of a triangle whose base is 16.75 feet, and height 6.24 feet? *fe.* *in.* *pa.*
Ans. 52 3. 1

4. Required the area of a triangle whose base is 12.25 chains, and height 8.5 chains. *ac.* *ro.* *po.*
Ans. 5 0 33

5. What is the area of a triangle whose base is 20 feet, and height 10.25? *Ans.* 102.5 *fe.*

PROBLEM III.

To find the area of a triangle whose three sides only are given.

RULE.*

1. From half the sum of the three sides subtract each side severally.

---

* *Demon.* Let AC$=a$, AB$=b$, BC$=c$, and AD$=x$; (See preceding fig.) Then, since BD$=b-x$, we shall have $c^2-(b-x)^2=$ CD$^2=a^2-x^2$, or $c^2-b^2+2bx-x^2=a^2-x^3$, from which x is found $=\frac{a^2+b^2-c^2}{2b}$ by trans. and reduction.

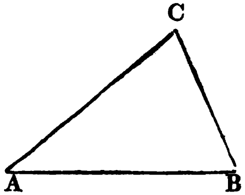
But CD^2 = AC^2 — AD^2 = $\overline{\text{AC}+\text{AD}}\times\overline{\text{AC}-\text{AD}}=\left(a+\frac{a^2+b^2-c^2}{2b}\right)\times\left(a-\frac{a^2+b^2-c^2}{2b}\right)=\frac{2ab+a^2+b^2-c^2}{2b}\left(\times\frac{2ab-a^2-b^3+c^2}{2b}\right)=\frac{(a+b)^2-c^2}{2b}\times\frac{c^2-(a-b)^2}{2b}$;

Whence CD$=\frac{1}{2b}\sqrt{\overline{a+}^2b-c^2)\times(c^2-\overline{a-b}^2)}$, and the area $\frac{1}{2}$ AB × CD $=\frac{1}{4}\sqrt{(\overline{a+b}^2-c^2)\times(c^2-\overline{a-b}^2)}$

2. Multiply the half sum and the three remainders continually together, and the square root of the product will be the area required.

EXAMPLES.

1. Required the area of the triangle ABC, whose three sides BC, CA, and AB are 24, 36, and 48 chains respectively.



Here $\frac{24+36+48}{2}=\frac{108}{2}=54=\frac{1}{2}$ *sum of the sides;*

$=\frac{1}{4}\sqrt{(a+b+c\times\overline{a+b-c}\times\overline{c+a-b}\times\overline{c-a+b})}$

$=\sqrt{(\frac{a+b+c}{2}\times\frac{a+b-c}{2}\times\frac{c+a-b}{2}\times\frac{c-a+b}{2})}$

which, by making $s=\frac{1}{2}\times(a+b+c)$ becomes $=$ $\sqrt{(s\times\overline{s-c}\times\overline{s-b}\times\overline{s-a})}$ $=$algebraical expression for the rule, as was to be demonstrated.

Cor. 1. If s be put equal to $a+b$, and $d=b\backsim a$, the rule is $\frac{1}{4}\sqrt{(s^2-c^2)\times(c^2-d^2)}$.

Cor. 2. If all the sides be equal, the rule will become $\frac{1}{4}a^2\sqrt{3}$, or $\frac{1}{4}a^2\times1.732$ for the equilateral triangle whose side is a.

Cor. 3. If the triangle be right angled, a being the hypothenuse, the rule will be $\frac{a+b+c}{2}\times\frac{-a+b+c}{2}$, or $\frac{1}{2}p\times\frac{1}{2}\overline{p-a}$, putting p for the perimeter.

Rule II. Any two sides of a triangle being multiplied together, and the product again by half the natural sine of their included angle, will give the area of the triangle.

That is, AC × CB × nat. sine of the angle C=twice area.

Also 54—24=30 *first diff.* 54—36=18 *second diff.* *and* 54—48=6 *third diff.*

Whence $\sqrt{54\times30\times18\times6} = \sqrt{174960} = 418.282 =$ *area required.*

2. Required the area of a triangle whose three sides are 13, 14 and 15 feet. *Ans.* 48 *square feet.*

3. How many acres are there in a triangle whose three sides are 49.00, 50.25 and 25.69 chains? *Ans.* 61.498 *ac.*

4. Required the area of a right angled triangle, whose hypothenuse is 50, and the other two sides 30 and 40. *Ans.* 600.

5. Required the area of an equilateral triangle, whose side is 25. *Ans.* 270.6329.

6. Required the area of an isosceles triangle, whose base is 20, and each of its equal sides 15. *Ans.* 111.803.

7. Required the area of a triangle, whose three sides are 20, 30, and 40 chains. *Ans.* 29 *ac.* 8 *po.*

PROBLEM IV.

Any two sides of a right angled triangle being given to find the third side.

RULE.*

1. *When the two legs are given to find the hypothenuse.*

Add the square of one of the legs to the square of the other, and the square root of the sum will be equal to the hypothenuse.

2. *When the hypothenuse and one of the legs are given to find the other leg.*

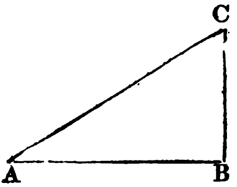
From the square of the hypothenuse take the square of the given leg, and the square root of the remainder will be equal to the other leg.

---

[*] By Euc. 47.I. $AB^2+BC^2=AC^2$, or $AC^2-AB^2=BC^2$; and therefore $\sqrt{AB^2+BC^2}=AC$, or $\sqrt{AC^2-AB^2}=BC$, or $\sqrt{AC^2-BC^2}=AB$, which *is the same as the rule.*

EXAMPLES.

1. In the right angled triangle ABC, the base AB is 56, and the perpendicular BC 33; what is the hypothenuse?



Here $56^2 + 33^2 = 3136 + 1089 = 4225$; *and* $\sqrt{4225} = 65$ $=$ *hypothenuse* AC.

2. If the hypothenuse AC be 53, and the base AB 45; what is the perpendicular BC?

Here $53^2 - 45^2 = 2809 - 2025 = 784$; *and* $\sqrt{784} = 28 =$ *perpendicular* BC.

3. The base of a right angled triangle is 77, and the perpendicular 36: what is the hypothenuse? *Ans.* 85.

4. The hypothenuse of a right angled triangle is 109, and the perpendicular 60: what is the base? *Ans.* 91.

5. It is required to find the length of a shore, which strutting 12 feet from the upright of a building, will support a jamb 20 feet from the ground. *Ans.* 23.3238 *feet.*

6. The height of a precipice, standing close by the side of a river, is 103 feet, and a line of 320 feet will reach from the top of it to the opposite bank: required the breadth of the river. *Ans.* 302.9703 *feet.*

7. A ladder 50 feet long, being placed in a street, reached a window 28 feet from the ground, on one side; and by turning it over, without removing the foot, it reached another window, 36 feet high, on the other side; required the breadth of the street. *Ans.* 76.1233 *feet.*

PROBLEM V.

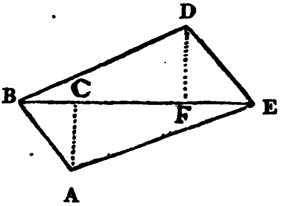
To find the area of a trapezium.

RULE.*

Multiply the diagonal by the sum of the two perpendiculars falling upon it from the opposite angles, and half the product will be the area.

EXAMPLES.

1. Required the area of the trapezium BAED, whose diagonal BE is 84, the perpendicular AC 21, and DF 28.



Here $\overline{28+21}\times 84=49\times 84=4116$; *and* $\frac{4116}{2}=2058$ *the area required.*

---

* *Demon.* The area of the triangle BDE is $=\frac{BE\times DF}{2}$; and the area of the triangle BAE is $=\frac{BE\times AC}{2}$; and therefore the sum of these areas, or the area of the whole trapezium, is $=\frac{BE\times DF}{2}+\frac{BE\times AC}{2}=\frac{DF+AC}{2}\times BE$. Q. E. D.

If the trapezium can be inscribed in a circle; that is, if the sum of two of its opposite angles is equal to two right angles, or 180°, the area may be found thus:

Rule. From half the sum of the four sides subtract each side severally; then multiply the four remainders continually together, and the square root of the product will be the area.

2. Required the area of a trapezium whose diagonal is 80.5, and the two perpendiculars 24.5 and 30.1.

Ans. 2197.65.

3. What is the area of a trapezium whose diagonal is 108 feet 6 inches, and the perpendiculars 56 feet 3 inches, and 60 feet 9 inches ? *Ans.* 6347 *fe.* 3 *in.*

PROBLEM VI.

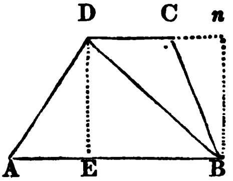
To find the area of a trapezoid, or a quadrangle, two of whose opposite sides are parallel.

RULE.*

Multiply the sum of the parallel sides by the perpendicular distance between them, and half the product will be the area.

EXAMPLES.

1. Required the area of the trapezoid ABCD, whose sides AB and DC are 321.51 and 214.24, and perpendicular DE 171.16.



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* *Demon.* The △ ABD is $=\frac{AB\times DE}{2}$, and the △ BCD$=\frac{DC\times Bn}{2}$, or, (because $Bn=DE$)$=\frac{DC\times DE}{2}$. Therefore, △ ABD$+$△BCD, or the whole trapezoid ABCD, is $=\frac{AB\times DE}{2}+\frac{DC\times DE}{2}=\frac{AB+DC}{2}\times DE$. Q. E. D.

Here $321.51+214.24=535.75=$ *sum of the parallel sides* AB, DC.

Whence 535.75×171.16 (*the perp.* DE) $=91698.9700$. *And* $\frac{91698.9700}{2}=45849.485$ *the area required.*

2. The parallel sides of a trapezoid are 12.41 and 8.22 chains, and the perpendicular distance 5.15 chains; required the area.
ac. ro. po.
Ans. 5 1 9.956.

3. Required the area of a trapezoid whose parallel sides are 25 feet 6 inches and 18 feet 9 inches, and the perpendicular distance 10 feet 5 inches.
fe. in. pa.
Ans. 230 5 7.

4. Required the area of a trapezoid whose parallel sides are 20.5 and 12.25, and perpendicular distance 10.75.
Ans. 176.03125.

PROBLEM VII.

To find the area of a regular polygon.

RULE.*

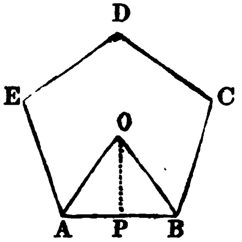
Multiply half the perimeter of the figure by the perpendicular falling from its centre upon one of the sides, and the product will be the area.

Note. The perimeter of any figure is the sum of all its sides.

---

[*] *Demon.* Every regular polygon is composed of as many equal triangles as it has sides, consequently the area of one of those triangles being multiplied by the number of sides must give the area of the whole figure; but the area of either of the triangles is equal to the rectangle of the perpendicular and half the base, and therefore the rectangle of the perpendicular and half the sum of the sides is equal to the area of the whole polygon; thus, $\mathrm{OP}\times\frac{\mathrm{AB}}{2}=$ area of the $\triangle$ AOB, and $\mathrm{OP}\times\frac{5\mathrm{AB}}{2}=$ area of the polygon ABCDE. Q. E. D.

1. Required the area of the regular pentagon ABCDE, whose side AB, or BC, &c. is 25 feet, and perpendicular OP 17.2 feet.



Here $\frac{25\times5}{2}=62.5=$ *half perimeter*; *and* $62.5\times17.2=$ 1075 *square feet= area required.*

2. Required the area of a hexagon whose side is 14.6 feet, and perpendicular 12.64.

Ans. 553.632 *square feet.*

3. Required the area of a heptagon whose side is 19.38, and perpendicular from the centre 20.

Ans. 1356.6.

4. Required the area of an octagon whose side is 9.941 and perpendicular 12. Ans. 477.168.

PROBLEM VIII.

To find the area of a regular polygon, when the side only is given.

RULE.*

Multiply the square of the side of the polygon by the number standing opposite to its name in the following table, and the product will be the area.

---

* *Demon.* The multipliers in the table are the areas of the polygons to which they belong when the side is unity, or 1.

Whence as all regular polygons, of the same number of sides,

No. of sides.	Names.	Multipliers.
3	Trigon or equil. △	0.433013—
4	Tetragon or square	1.000000+
5	Pentagon	1.720477+
6	Hexagon	2.598076+
7	Heptagon	3.633912+
8	Octagon	4.828427+
9	Nonagon	6.181824+
10	Decagon	7.694209—
11	Undecagon	9.365640—
12	Duodecagon	11.196152+

are similar to each other, and as similar figures are as the squares of their like sides, (*Euc.* VI. 20.) 1^2: multiplier in the table:: square of the side of any polygon: area of that polygon; or which is the same thing, the square of the side of any polygon × by its tabular number is = area of the polygon. Q. E. D.

The table is formed by trigonometry, thus: As radius = 1 : tang.

$$\angle \text{OBP} : : \text{BP}\ (\tfrac{1}{2}) : \text{PO} = \frac{\text{BP} \times \text{tan.} \angle \text{OBP}}{\text{radius}} = \tfrac{1}{2}\ \text{tang.}\ \angle\ \text{OBP} :$$

whence OP×BP = ¼ tang. ∠ OBP = area of the △ AOB; and ¼ tang. ∠ OBP × number of sides = tabular number, or the area of the polygon.

The angle OBP, together with its tangent, for any polygon of not more than 12 sides, is shown in the following table.

No. of sides.	Names.	Angle OBP.	Tangents.
3	Trigon	30°	$.57735+=\frac{1}{3}\sqrt{3}$
4	Tetragon	45°	$1.00000+=1\times 1$
5	Pentagon	54°	$1.37638+=\sqrt{1+\frac{2}{5}\sqrt{5}}$
6	Hexagon	60°	$1.73205+=\sqrt{3}$
7	Heptagon	$64^\circ\frac{2}{7}$	$2.07652+$
8	Octagon	$67^\circ\frac{1}{2}$	$2.41421+=1+\sqrt{2}$
9	Nonagon	70°	$2.74747+$
10	Decagon	72°	$3.07768+=\sqrt{5+2\sqrt{5}}$
11	Undecagon	$73^\circ\frac{7}{11}$	$3.40568+$
12	Duodecagon	75°	$3.73205+=2+\sqrt{3}$

EXAMPLES.

1. Required the area of a pentagon whose side is 15.

The number opposite pentagon in the table is 1.720477.

Hence $1.720477 \times 15^2 = 1.720477 \times 225 = 387.107325 =$ *area required.*

2. The side of a hexagon is 5 feet 4 inches; what is the area? *Ans.* 73.9.

3. Required the area of an octagon whose side is 16. Ans. 1236.0773.

4. Required the area of a decagon whose side is 20.5. Ans. 3233.4913.

5. Required the area of a nonagon whose side is 36. Ans. 8011.6439.

6. Required the area of a duodecagon whose side is 125. Ans. 174939.875.

PROBLEM IX.

The diameter of a circle being given, to find the circumference; or, the circumference being given, to find the diameter.

RULE.*

Multiply the diameter by 3.1416, and the product will be the circumference, or

---

* The proportion of the diameter of a circle to its circumference has never yet been exactly ascertained. Nor can a square or any other right lined figure, be found, that shall be equal to a given circle. This is the celebrated problem called the squaring of the circle, which has exercised the abilities of the greatest mathemati-

Divide the circumference by 3.1416 and the quotient will be the diameter.

Note 1.—As 7 is to 22, so is the diameter to the circumference; or as 22 is to 7, so is the circumference to the diameter.

2. As 113 is to 355, so is the diameter to the circumference; or, as 352 is to 115, so is the circumference to the diameter.

---

cians for ages, and been the occasion of so many disputes. Several persons of considerable eminence, have, at different times, pretended that they had discovered the exact quadrature; but their errors have soon been detected, and it is now generally looked upon as a thing impossible to be done.

But though the relation between the diameter and circumference cannot be accurately expressed in known numbers, it may yet be approximated to any assigned degree of exactness. And in this manner was the problem solved by the great Archimedes, about two thousand years ago, who discovered the proportion to be nearly as 7 to 22, which is the same as our first note. This he effected by showing that the perimeter of a circumscribed regular polygon of 192 sides, is to the diameter in a less ratio than that of $3\frac{10}{70}$ to 1, and that the perimeter of an inscribed polygon of 96 sides is to the diameter in a greater ratio than that of $3\frac{10}{71}$ to 1, and from thence inferred the ratio above mentioned, as may be seen in his book *De Dimensione Circuli.* The same proportion was also discovered by Philo Gedarensis and Apollonius Pergeus at a still earlier period, as we are informed by Eutocius in his observations on a work called Ocyteboos.

The proportion of Vieta and Metius is that of 113 to 355, which is something more exact than the former, and is the same as the second note.

This is a very commodious proportion: for being reduced into decimals, it agrees with the truth as far as the sixth figure inclusively. It was derived from the pretended quadrature of a M. Van Eick, which first gave rise to the discovery.

But the first who ascertained this ratio to any great degree of exactness was Van Ceulen, a Duchman, in his book, *De Circulo et Adscriptis.* He found that if the diameter of a circle was 1, the circumference would be 3.141592653589793238462643383279502884 nearly; which is exactly true to 36 places of decimals, and was effected by the continual bisection of an arc of a circle, a method so

EXAMPLES.

1, If the diameter of a circle be 17, what is the circumference?

Here $3.1416 \times 17 = 53.4072 =$ *circumference.*

2. If the circumference of a circle be 354, what is the diameter?

Here $\frac{354.000}{3.1416} = 112.681 =$ *diameter.*

3. What is the circumference of a circle whose diameter is 40 feet? Ans. 125.6640.

4. What is the circumference of a circle whose diameter is 12 feet? Ans. 37.6992.

5. If the circumference of the earth be 25000 miles, what is its diameter? Ans. 7958 *nearly.*

6. The base of a cone is a circle; what is its diameter when the circumference is 54 feet? Ans. 20.3718.

---

extremely troublesome and laborious that it must have cost him incredible pains. It is said to have been thought so curious a performance, that the numbers were cut on his tomb-stone in St. Peters Church-yard, at Leyden. This last number has since been confirmed and extended to double the number of places, by the late ingenious Mr. Abraham Sharp, of Little Horton, near Bedford, in Yorkshire.

But since the invention of Fluxions, and the Summation of Infinite Series, there have been several methods discovered for doing the same thing with much more ease and expedition. The late Mr. John Machin, Professor of Astronomy in Gresham College, has by these means given a quadrature of the circle which is true to 100 places of decimals; and M. de Lagny, M. Euler, &c. have carried it still further. All of which proportions are so extremely near the truth, that, except the ratio could be completely obtained, we need not wish for a greater degree of accuracy.

PROBLEM X.

To find the length of any arc of a circle.

RULE.*

1 *When the chord of the arc and the versed sine of half the arc are given.*

To 15 times the square of the chord, add 33 times the square of the versed sine,† and reserve the number.

To the square of the chord, add 4 times the square of the versed sine, and the square root of the sum will be *twice the chord of half the arc.*

Multiply twice the chord of half the arc by 10 times

---

* *Demon.* Put $c = \frac{1}{2}$ the chord of the arc, and $v =$ the versed sine of half the arc, then the rule may be expressed thus:

$$\sqrt{(4c^2+4v^2)} + \frac{\sqrt{(4c^2+v^2)}.10v^2}{60c^2+33v^2} = 2\sqrt{(c^2+v^2)}.\left(1+\frac{10v^2}{60c^2+33v^2}\right) = 2\sqrt{\left(\frac{c^2+v^2}{v}.v\right)}.\left(1+\frac{10v}{60\frac{c^2+v^2}{v}-27v}\right) =$$

$$2\sqrt{dv}\left(1+\frac{10v}{60d-27v}\right) = 2\sqrt{dv}\left(1+\frac{v}{6d}+\frac{3v^2}{40d^2}+\frac{27v^3}{800d^3}\right)$$

Now $2\sqrt{dv}.(1+\frac{v}{6d}+\frac{3v^2}{40d^2}+\frac{5v^3}{112d^3}+$ &c.) is known to be the length of an arc whose diameter is d, and the versed sine of half the arc v; and this differs from the preceding only by $\frac{61v^3}{5600d^3}$ &c.

† Here, as in many other places in the following part of the work, the term *versed sine* is used instead of *versed sine of half the arc;* but in all cases of the kind, it is the versed sine of half the arc that is to be understood.

the square of the versed sine, divide the product by the reserved number, and add the quotient to twice the chord of half the arc: the sum will be the length of the arc very nearly.

When the chord of the arc, and the chord of half the arc are given.—From the square of the chord of half the arc subtract the square of half the chord of the arc, the remainder will be the square of the versed sine: then proceed as above.

2. *When the diameter and the versed sine of half the arc are given.*

From 60 times the diameter subtract 27 times the versed sine, and reserve the number.

Multiply the diameter by the versed sine and the square root of the product will be the *chord of half the arc.*

Multiply twice the chord of half the arc by 10 times the versed sine, divide the product by the reserved number, and add the quotient to twice the chord of half the arc; the sum will be the length of the arc very nearly.

Note 1.—When the diameter and chord of the arc are given, the versed sine may be found thus: From the square of the diameter subtract the square of the chord, and extract the square root of the remainder. Subtract this root from the diameter, and half the remainder will give the versed sine of half the arc.

2. The square of the chord of half the arc being divided by the diameter will give the versed sine, or being divided by the versed sine will give the diameter.

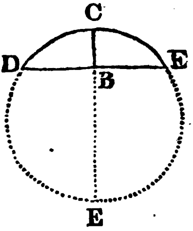
3. The length of the arc may also be found by multiplying together the number of degrees it contains, the radius and the number .01745329.

Or, as 180 is to the number of degrees in the arc, so is 3.1416 times the radius, to the length of the arc.

Or, as 3 is to the number of degrees in the arc, so is .05236 times the radius, to the length of the arc.*

EXAMPLES.

1. If the chord DE be 48, and the versed sine CB 18, what is the length of the arc? Ans. 64.2959.



Here $48^2 \times 15 = 34560$

$18^2 \times 33 = 10692$

45252 *reserved number.*

$48^2 = 2304 =$ *the square of the chord.*

$18^2 \times 4 = 1296 = 4$ *times the square of the versed sine.*

$\sqrt{3600} = 60 =$ *twice the chord of half the arc* ACB.

Now $\frac{60 \times 18^2 \times 10}{45252} = \frac{194400}{45252} = 4.2959$, *which added to twice the chord of half the arc gives* 64.2959 = *the length of the arc.*

2. Given the diameter CE 50, and the versed sine CD 18, what is the length of the arc? Ans. 64.2959.

---

* When very great accuracy is required, the following theorem may be used. Let d denote the diameter of the circle, and v the versed sine of half the arc, then the arc $= 2\sqrt{dv} \times \left(1 + \frac{v}{6d} + \frac{3v^2}{40d^2} + \frac{5v^3}{112d^3} + \frac{35v^4}{1152d^4} + \frac{63v^5}{2816d^5} + \text{, \&c.}\right)$

$$\begin{array}{r} 50\times60=3000 \\ 18\times27=486 \\ \hline 2514 \end{array}$$ *reserved number.*

AC$=\sqrt{\overline{50\times18}}=30=$*the chord of half the arc.*

$\frac{\overline{30\times2}\times\overline{18\times10}}{2514}=\frac{10800}{2514}=4.2959$, *which added to twice the chord of half the arc gives* 64.2959*=the length of the arc* ACB.

3. The chord of the whole arc is 7, and the versed sine 2, what is the length of the arc? Ans. 8.4343.

4. The chord of the whole arc is 40, and the versed sine 15, what is the length of the arc? Ans. 53.5800.

5. The chord of the whole arc is 50, and the chord of half the arc 27, required the length of the arc. Ans. 55.3720.

6. Given the diameter of the circle 100, and the versed sine 9, required the length of the arc. Ans. 60.9380.

7. Given the chord of the whole arc 16, and the diameter of the circle 20, required the length of the arc. Ans. 18.5439.

8. The diameter of the circle is 50, and the chord of half the arc 30, what is the length of the arc? Ans. 64.2959.

9. The chord of half the arc is 25, and the versed sine 15, required the length of the arc. Ans. 53.5800.

PROBLEM XI.

To find the area of a circle.

RULE I.*

Multiply half the circumference by half the diameter, and the product will be the area.

---

* *Demon.* A circle may be considered as a regular polygon of an infinite number of sides, the circumference being equal to the perimeter, and the radius to the perpendicular. But the area of a regular polygon is equal to half the perimeter multiplied by the

Or take $\frac{1}{4}$ of the product of the whole circumference and diameter.

EXAMPLES.

1. What is the area of a circle whose diameter is 42, and circumference 131.946?

2)131.946

65.973 = $\frac{1}{2}$ *cirumference.*
21 = $\frac{1}{2}$ *diameter.*

65973
131946

1385.433 = *area required.*

2. What is the area of a circle whose diameter is 10 feet 6 inches, and circumference 31 feet 6 inches?

fe. *in.*
15 9 = 15.75 = $\frac{1}{2}$ *circumference.*
5 3 = 5.25 = $\frac{1}{2}$ *diameter.*

7875
3150
7875

82.6875
12

8.2500

Ans. 82 *feet* 8 *inches.*

---

perpendicular, and consequently the area of a circle is equal to half the circumference multiplied by the radius, or half the diameter. Q. E. D.

This rule may be otherwise demonstrated by the doctrine of fluxions.

3. What is the area of a circle whose diameter is 1, and circumference 3.1416? Ans. .7854.

4. What is the area of a circle whose diameter is 7, and circumference 22? Ans. 38½.

RULE II.*

Multiply the square of the diameter by .7854, and the product will be the area; or,

Multiply the square of the circumference by .07958 and the product will be the area.

---

* *Demon.* All circles are to each other as the squares of their diameters. (Euc. XII. 2.)

But the area of a circle whose diameter is 1, is .7854, &c. (by Rule I.) Therefore $1^2 : d^2 :: .7854, \&c. : \frac{.7854, \&c. \times d^2}{1} =$ $.785, \&c. \times d^2 =$ area of a circle whose diameter is d. Q. E. D.

The following proportions are those of *Metius* and *Archimedes*.

As 452 : 355 : : square of the diameter : area.

As 14 : 11 : : square of the diameter : area.

If the circumference be given, instead of the diameter, the area may be found as follows:

The square of the circumference ×.07958=area.

As 88 : 7 : : square of the circumference : area.

As 1420 : 113 : : square of the circumference : area.

And if d be the diameter, c the circumference, a the area, and p =3.14159, &c. then:

$$1.\ d = \frac{c}{p} = \frac{4a}{c} = 2\sqrt{\frac{a}{p}}$$

$$2.\ c = pd = \frac{4a}{d} = 2\sqrt{pa}$$

$$3.\ a = \frac{pd^2}{4} = \frac{c^2}{4p} = \frac{dc}{4}$$

The following table will also show most of the useful problems relating to the circle and its equal or inscribed square.

1. diameter × .8862 = side of an equal square.
2. circumf. × .2821 = side of an equal square.
3. diameter × .7071 = side of the inscribed square.
4. circum. × .2251 = side of the inscribed square.

EXAMPLES.

1. What is the area of a circle whose diameter is 5?

```
    .7854
       25=square of the diameter.
   ------
    39270
   15708
   ------
   19.6350=the answer.
```

2. What is the area of a circle whose diameter is 7? Ans. 38.4846.

3. What is the area of a circle whose diameter is 4.5? Ans. 15.9043.

4. How many square yards are there in a circle whose diameter is 3½ feet? Ans. 1.0690.

5. How many square feet are there in a circle whose circumference is 10.9956 yards? Ans. 86.5933.

6. How many square perches are there in a circle whose circumference is 7 miles? Ans. 399300.6080.

PROBLEM XII.

To find the area of a sector, or that part of a circle which is bounded by any two radii and their included arc.

RULE.*

Find the length of the arc by Problem X. then multiply the radius, or half the diameter, by the length of the arc of the sector, and half the product will be the area.

---

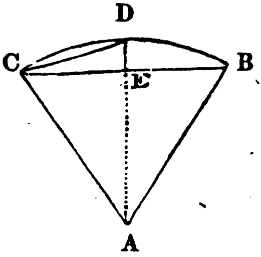
5. area × .6366=side of the inscribed square.
6. side of a square × 1.4142=diam. of its circums. circle.
7. side of a square × 4.443=circumf. of its circums. circle.
8. side of a square × 1.128=diameter of an equal circle.
9. side of a square × 3.545=circumf. of an equal circle.

* The rule for finding the area of the sector, is evidently the same as that for finding the area of the whole circle.

Note.—If the diameter or radius is not given, add the square of half the chord of the arc, to the square of the versed sine of half the arc; this sum being divided by the versed sine, will give the diameter.

EXAMPLES.

1. The radius AB is 40, and the chord BC of the whole arc 50, required the area of the sector.



$\frac{80-\sqrt{80^2-50^2}}{2}=8.7750=$ *the versed sine of half the arc.*

$\overline{80\times60-8.7750\times27}=4563.0750=$ *the reserved number.*

$2\times\sqrt{8.7750\times80}=52.9906=$ *twice the chord of half the arc.*

$\frac{52.9906\times8.7750\times10}{4563.0750}=1.0190$ *which added to twice the chord of half the arc gives* 54.0096 *the length of the arc.*

And $\frac{54.0096\times40}{2}=1080.1920=$ *area of the sector required.*

2. Find the area of the sector, the chord of whose arc is 40, and the versed sine of half the arc 15.

Ans. 558.1250.

3. Required the area of the sector, the chord of half the arc being 30, and the diameter of the circle 100.

Ans. 1523.4500.

4. Given the diameter of the circle 50, and the versed sine 18, to find the area of the sector. Ans. 803.69875.

RULE II.*

As 360 is to the degrees in the arc of a sector, so is the area of the whole circle, whose radius is equal to that of the sector, to the area of the sector required.

Note—For a semicircle, a quadrant, &c. take one half, one quarter, &c. of the whole area.

EXAMPLES.

1. The radius of a sector of a circle is 20, and the degrees in its arc 22; what is the area of the sector?

Here the diameter is 40.

Hence, by Rule II. Prob. XII. *the area of the circle*$=40^2\times.7854=1600\times.7854=1256.64$.

Now, $360^\circ : 22^\circ :: 1256.64 : 76.7947=$ *area of the sector*.

---

* *Demon.* Let $r=$ radius, $d=$ number of degrees in the arc of the sector, and $A=$ its area.

Then will $4r^2\times.7854=r^2\times3.1416=$ area of the whole circle, and $2r\times3.1416=$ its circumference.

Also $360 : 2r\times3.1416 :: d : \frac{2dr\times3.1416}{360}=$ length of the arc of the sector. But $\frac{2dr\times3.1416}{360}\times\frac{1}{2}\times r=\frac{dr^2\times3.1416}{360}$ $=A$, by the last rule. And consequently $360 : d :: r^2\times3.1416 : A$. Q. E. D.

2. Required the area of a sector whose radius is 25, and the length of its arc 147 degrees 29 minutes.
Ans. 804.3987.

3. Required the area of a semicircle whose radius is 13.
Ans. 265.4652.

4. Required the area of a quadrant whose radius is 21.
Ans. 346.3614.

PROBLEM XIII.

To find the area of a segment of a cirble.

RULE I.*

1. Find the area of the sector, having the same arc with the segment, by the last problem.

2. Find the area of the triangle formed by the chord of the segment, and the radii of the sector.

3. Then the sum, or difference, of these areas, according as the segment is greater or less than a semicircle, will be the area required.

Note.—The difference between the versed sine and radius, multiplied by half the chord of the arc, will give the area of the triangle.

EXAMPLES.

1. The radius OB is 10, and the chord AC 10; what is the area of the segment ABC?

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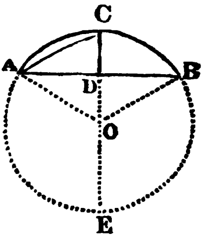
* This rule is too evident to need a demonstration.



$CD=\frac{AC^2}{CE}=\frac{100}{20}=5=$ *the versed sine of half the arc.*

$\overline{20\times60}-\overline{5\times27}=1065=$ *the reserved number.*

$\frac{\overline{10\times2}\times\overline{5\times10}}{1065}=.9390$, *and this added to twice the chord of half the arc gives* $20.9390=$ *the length of the arc.*

$\frac{20.9390\times10}{2}=104.6950=$ *area of the sector* OACB.

$OD=OC-CD=5$ *the perpendicular height of the triangle.*

$AD=\sqrt{\overline{AO^2-OD^2}}=\sqrt{\overline{75}}=8.6603=\frac{1}{2}$ *the chord of the arc.*

$8.6603\times5=43.3015=$ *the area of the triangle* AOB.

$104.6950-43.3015=61.3935=$ *area of the segment required;* it being in this case less than a semicircle.

2. Required the area of a segment whose height is 2, and chord 20. Ans. 26.8783.

3. Required the area of a segment of a circle, the radius being 10, and the chord of the arc 12. Ans. 16.3500.

4. Required the area of a segment of a circle, whose chord is 16, and the diameter of the circle 20.
Ans. 44.7195.

5. What is the area of a segment whose arc is a quadrant, the diameter of the circle being 18 feet?
Ans. 23.1174.

6.* What is the area of a segment, whose arc contains 300 degrees, the diameter being 50? Ans. 1906.8831.

RULE II.†

1. Divide the height, or versed sine, by the diameter, and find the quotient in the table of versed sines.

2. Multiply the number on the right hand of the versed sine by the square of the diameter, and the product will be the area.

Note 1.—When the diameter or versed sine is not given, it may be found by the notes, page 68 or 74.

Note 2.—When the quotient arising from the versed sine divided by the diameter, has a remainder or fraction after the third place of decimals; having taken the area

---

* The chord of the arc will evidently be equal to the radius of the circle.

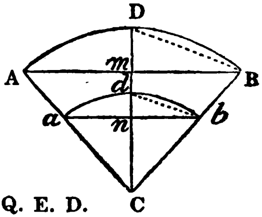
† The table to which this rule refers, is formed of the areas of the segments of a circle whose diameter is 1; and which is supposed to be divided by perpendicular chords into 1000 equal parts.

The reason of the rule itself depends upon this property—That the versed sines of similar segments are as the diameters of the circles to which they belong, and the areas of those segments as the squares of the diameters; which may be thus demonstrated.

Let ADBA and *adba* be any two similar segments, cut off from the similar sectors ADBCA and *adbca*, by the chords AB and *ab*, and let the perpendicular CD bisect them.

Then by similar triangles, DB : *db* : : BC : *bc* and DB : *db* : : D*m* : *dn*; whence, by equality, BC : *bc* : : D*m* : *dn*, or 2BC : 2*bc* : : D*m* : *dn*.

Again, since similar segments are as the squares of their chords, it will be $AB^2 : ab^2$: : seg. ADBA : seg. *adba*; but $AB^2 : ab^2 :: CB^2 : cb^2$, whence, by equality, seg. ABDA : seg. *adba* : : $CB^2 : cb^2$, or seg. ADBA : seg. *adba* : : $4CB^2 : 4cb^2$. Q. E. D.



Now, If *d* be put equal to any diameter, and *v* the versed sine, it will be $d : v :: 1$ (diameter in the table):$\frac{v}{d}$=versed sine of a similar segment in the table, whose area let be called *a*.

answering to the first three figures, subtract it from the next following area, multiply the remainder by the said fraction, and add the product to the first area, then the sum will be the area for the whole quotient.

EXAMPLES.

1. If the chord of a circular segment be 40, its versed sine 10, and the diameter of the circle 50, what is the area?

5.0)1.0
.2= *tabular versed sine.*
.111823= *tabular segment.*
2500= *square of 50.*
55911500
223646
279.557500= *area required.*

2. The chord of the segment is 20, the versed sine 5, what is the area? Ans. 69.889375.

3. The diameter of a circle is 40, and the versed sine 10; what is the area of a segment? Ans. 245.6736.

4. If the diameter be 52, and the versed sine 2, what is the area of the segment? Ans 26.9197.

5. If the chord of half the arc be 30, and versed sine 9, what is the area of the segment? Ans. 350.1100.

6. The diameter of a circle is 100, and the chord of the arc 60: what is the area of the segment? Ans. 408.75.

Then $1^2 : d^2 :: a : ad^2$=area of the segment whose height is v, and diameter d as in the rule.

PROBLEM XIV.

To find the area of a circular zone, or the space included between any two parallel chords and their intercepted arcs.

RULE.*

From the greater chord subtract half the difference between the two, multiply the remainder by the said half difference, divide the product by the breadth of the zone, and add the quotient to the breadth. To the square of this number add the square of the less chord, and the square root of the sum will be the diameter of the circle.

Now, having the diameter EG, and the two chords AB and DC, find, by Prob. XIII. Rule II. the areas of the segments ABEA, and DCED, the difference of which will be the area of the zone required.

Note 1.—The difference of the tabular segments multiplied by the square of the circle's diameter will give the area of the zone.

Note 2.—When the larger segment AEB is greater than a semicircle, find the areas of the segments AGB and DCE, and subtract their sum from the area of the whole circle, the remainder will be the area of the zone.

---

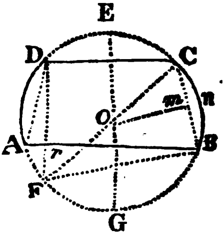
* The reason of this rule is too obvious to require a demonstration.

Note.—When the two parallel sides of the zone are equal, the chord of the small segment will be equal to the breadth of the zone, and the height of this segment will be equal to $\sqrt{\frac{1}{4}AB^2+\frac{1}{4}as^2}=\frac{1}{2}AB$; *as* being put for the breadth of the zone.

And when one of the sides is the diameter of the circle, the chord of the same segment will be $\sqrt{as^2+D^2}$, and its height $=\frac{1}{2}AB-\sqrt{\frac{1}{4}AB^2-\frac{1}{4}as^2-\frac{1}{4}D^2}$ where $D=\frac{1}{2}\overline{AB-DC}$.

EXAMPLES.

1. The greater chord AB is 20, the less DC 15, and their distance Dr $17\frac{1}{2}$: required the area of the zone ABCD. Ans. 395.4388.



$\frac{20-15}{2}=2.5=\frac{1}{2}$ *the difference between the two chords.*

$17.5+\frac{(20-2.5)\times 2.5}{17.5}=17.5+2.5=20=$ DF.

And $\sqrt{20^2+15^2}=\sqrt{625}=25$ =*the diameter of the cir.*

The segment AEB *being greater than a semicircle, we find by note* 1, *page* 68, *the versed sine of* DCE=2.5, *and that of* AGB=5.

Hence by Prob. XIII. Rule II., $\frac{2.5}{25}=.100=$ *tabular versed sine of* DEC.

And $\frac{5}{25}=.200=$ *tabular versed sine of* AGB.

Now $.040875\times 25^2=$ *area of seg.* DEC $=25.546875$
And $.111823\times 25^2=$ *area of seg.* AGB $=69.889375$

sum 95.43625

By Prob. XI. Rule II. $.7854\times 25^2$ =*area of the whole circle* } $=490.87500$

Difference=area of the zone ABCD $=395.43875$

2. The greater chord is 96, the lesser 60, and the breadth 26; what is the area of the zone?

Ans. 2136.7500.

3. One end of a circular zone is 48, the other end is 30, and the breadth is 13; what is the area of the zone?

Ans. 534.1875.

PROBLEM XV.

To find the area of a circular ring, or the space included between the circumferences of two concentric circles.

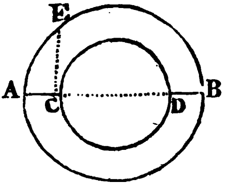
RULE.*

The difference between the areas of the two circles will be the area of the ring.

Or, multiply the sum of diameters by their difference, and this product again by .7854, and it will give the area required.

EXAMPLES.

1. The diameters AB and CD are 20 and 15; required the area of the circular ring, or the space included between the circumferences of those circles.



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* *Demon.* The area of the circle $AEBA = AB^2 \times .7854$, and the area of the small circle CD is $= CD^2 \times .7854$; therefore the area of the ring $= AB^2 \times .7854 - CD^2 \times .7854 = \overline{AB + CD} \times \overline{AB - CD} \times .7854$. Q. E. D.

Coroll. If CE be a perpendicular at the point C, the area of the ring will be equal to that of a circle whose radius is CE.

Rule 2. Multiply half the sum of the circumferences by half the difference of the diameters, and the product will be the area.

This rule will also serve for any part of the ring, using half the sum of the intercepted arcs for half the sum of the circumferences.

Here $\overline{AB+CD}\times\overline{AB-GD}=35\times5=175$; *and* $175\times.7854=137.4450=$ *area of the ring required.*

2. The diameters of two concentric circles are 16 and 10; what is the area of the ring formed by those circles? Ans. 122.5224.

3. The two diameters are 21.75 and 9.5, required the area of the circular ring. Ans. 300.6609.

4. Required the area of the ring, the diameters of whose bounding circles are 6 and 4. Ans. 15.708.

PROBLEM XVI.

To find the areas of lunes, or the spaces between the intersecting arcs of two eccentric circles.

RULE.*

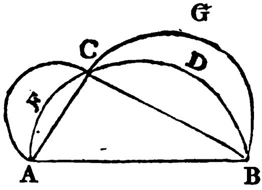
Find the areas of the two segments from which the lune is formed, and their difference will be the area required.

---

[*] Whoever wishes to be acquainted with the properties of lunes, and the various theorems arising from them, may consult *Mr. Wiston's Commentary on Tacquet's Euclid*, where they will find this subject very ingeniously managed.

The following property is one of the most curious:

If ABC be a right angled triangle; and semicircles be described on the three sides as diameters, then will the said triangle be equal to the two lunes D and F taken together.

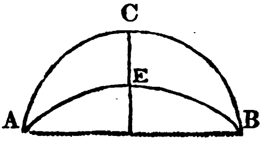


For the semicircles described on AC and BC=the one described on AB (31.66,) from each take the segments cut off by AC and BC, then will the lunes AFCE and BDCG=the triangle ACB.

Q. E. D.

EXAMPLES.

The length of the chord AB is 40, the height DC 10, and DE 4; required the area of the lune ACBEA.



By note, page 74, the diameter of the circle of which ACB *is a part*$=\frac{20^2+10^2}{10}=50$.

And the diameter of the circle of which AEB *is a part* $=\frac{20^2+4^2}{4}=104$.

Now, having the diameter and versed sines, we find by Prob. XIII. Rule III.

The area of seg. ACB$=.111823\times 50^2=279.5575$
And area of seg. AEB$=.009955\times 104^2=107.6733$

Their difference is the area of the lune AEBCA, *required,* $\Big\} = 171.8842$

2. The chord is 20, and the heights of the segment 10 and 2; required the area of the lune. Ans. 130.1599.

3. The length of the chord is 48, and the heights of the segments 18 and 12; required the area of the lune. Ans. 233.8122.

PROBLEM XVII.

To find the area of an irregular polygon, or a figure of any number of sides.

RULE.*

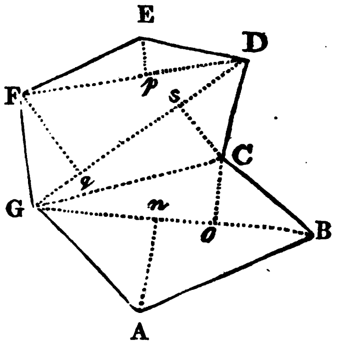
1. Divide the figure into triangles and trapeziums, and find the area of each separately.

2. Add these areas together, and the sum will be equal to the area of the whole polygon.

EXAMPLES.

1. Required the area of the irregular figure ABCDEFGA, the following lines being given.

$GB=30.5$ $An=11.2$, $CO=6$
$GD=29$ $Fq=11$ $Cs=6.6$
$FD=24.8$ $Ep=4$ ......



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* When any part of the figure is bounded by a curve, the area may be found as follows:

Rule 1. Erect any number of perpendiculars upon the base, at equal distances, and find their lengths.

2. Add the lengths of the perpendiculars, thus found, together, and the sum divided by their number will give the *mean breadth*.

3. Multiply the mean breadth by the length of the base, and it will give the area of that part of the figure required.

Here $\frac{An+Co}{2}\times GB=\frac{11.2+6}{2}\times 30.5=8.6\times 30.5=262.3$ *= area of the trapezium* ABCG.

And $\frac{Fq+Cs}{2}\times GD=\frac{11+6.6}{2}\times 29=8.8\times 29=255.2$ *= area of the trapezium* GCDF.

Also $\frac{FD\times Ep}{2}=\frac{24.8\times 4}{2}=\frac{99.2}{2}=49.6$ *= area of the triangle* FDE.

Whence 262.3+255.2+49.6=567.1 *= area of the whole figure required.*

2. *In a pentangular field, beginning with the south side and measuring round towards the east, the first, or south side, is 2735 links, the second 3115, the third 2370, the fourth 2925, and the fifth 2220; also the diagonal from the first angle to the third is 3800 links, and that from the third to the fifth 4010; required the area of the field.

Ans. 117 ac. 2 ro. 39 po.

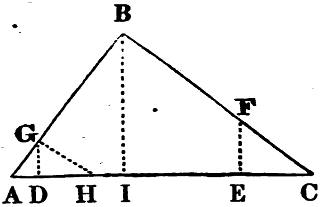
Promiscuous Questions concerning Lines and Areas.

1. Given AC=32, AD=3, EC=8, the perpendicular DG=4, and the perpendicular EF=6, to find the area of the triangle ABC and the sides AB and BC.

To find the area of mixed or compound figures, or such as are composed of rectilineal and curvilineal figures together; the rule is to find the area of the several figures of which the whole figure is composed, then add all the areas together, and the sum will be the area of the whole compound figure. And in the same manner may any irregular field or piece of land be measured, by dividing it into trapeziums and triangles, and finding the area of each separately.

* *Note.* As this figure consists of three triangles, all of whose sides are given, by calculating their areas according to Problem III. the sum will be the area of the whole figure accurately, without drawing perpendiculars from the angles to the diagonals.

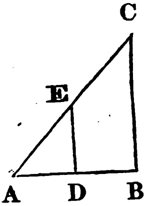
The same thing may also be done in most other cases of this kind.



Draw GH parallel to BC, then since the sides about the equal angles in equiangular or similar triangles are proportional, we have in the similar triangles CEF and HDG, FE : EC : : GD : DH $=\frac{16}{3}$, which added to AD will give AH $=\frac{25}{3}$. Also in the sim. △'s AGH, ABC, AH : AC : : GD : BI=15.36, and this multiplied by half the base will give the area of ABC=245.76.

Again in each of the right angled triangles ADG and CEF we have the two legs to find the hypothenuses AG= 5 and CF=10. Now by sim. △'s ADG, AIB, GD : GA : : BI : BA=19.2; and FE : FC : : BI : BC.

2. If from the right angled triangle ABC, whose base is 12, and perpendicular 16 feet, be cut off, by a line DE parallel to the perpendicular, a triangle whose area is 24 square feet; what are the sides of this triangle?



The area of the triangle ABC=AB×½BC=96, also having AB and BC, AC may be found=20.

Now it is evident that the triangles ABC and ADE ar similar, and since the areas of sim. △'s are as the square of their like sides, we have,

Area ABC : *area* ADE : : AC^2 : AE^2
Area ABC : *area* ADE : : BC^2 : DE^2
Area ABC : *area* ADE : : AB^2 : AD^2
From which $AE=10$, $AB=6$, and $DE=8$. Ans.

3. A gentleman in his yard has a circular grass plot, the diameter of which is 25 yards. Query the length of the string that would describe a circle to contain nine times as much. Ans. 37.5 *yards*.

4. Suppose a ladder 100 feet long, placed against a perpendicular wall 100 feet high, how far would the top of the ladder move down the wall by pulling out the bottom thereof 10 feet? Ans. .5012563.

5. *There is a circular pond whose area is $5028\frac{4}{7}$ square feet, in the middle of which stood a pole 100 feet high, now the pole having been broken, it was observed that the top just struck the brink of the pond; what is the height of the pole? Ans. 41.9968.

6. †In a level garden there are two lofty firs, having their tops ornamented with gilt balls, one is 100 feet high, the other 80, and they are 120 feet distant at the bottom; now the owner wants to place a fountain in a right line between the trees, to be equally distant from the top of each; what will be its distance from the bottom of each tree, and also from each of the balls?

Ans. { From the bottom of the lower tree 74 *feet*.
From the bottom of the higher tree 45 *feet*.
From each ball 109.6585 *feet*.

---

* This problem may be constructed by forming a right angled triangle, having the radius of the circle for the base, and the length of the pole for the perpendicular; and erecting a perpendicular on the middle of the hypothenuse to cut the perpendicular of the triangle; this will determine the place where the pole was broken.

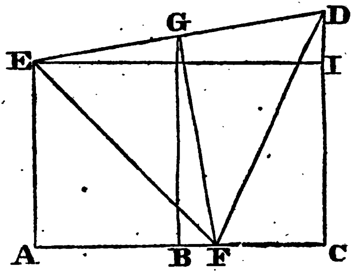
† The figure to this question is thus constructed. Draw $AC=120$, the distance of the trees at the bottom, and erect the perpen-

7. A person wishes to enclose 6*ac.* 1*ro.* 12*po.* in a triangle similar to a small triangle whose sides are 9, 8, and 6 perches respectively; required the sides of the triangle.
Ans. 59.029, 52.47, and 39.353 *perches.*

8. Required the sides of an isosceles triangle, containing 6*ac.* 0*ro.* 12*per.* and whose base is 72 perches.
Ans. 45 *perches* each.

---

diculars AE=height of the lower tree, and CD=the higher. Join ED, and from the middle of it draw the perpendicular GF, and F will represent the place of the fountain. Join EF and DF, and draw EI parallel to AC, and GB parallel to DC; then the triangles EID and GBF being similar, the calculation is evident.



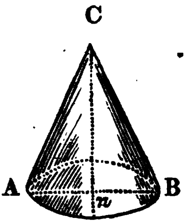
OF THE

CONIC SECTIONS.

DEFINITIONS.

1. The *conic sections* are such plain figures as are formed by the cutting of a cone.

2. *A *cone* is a solid described by the revolution of a right-angled triangle about one of its legs, which remains fixed.



3. The *axis of the cone* is the right line about which the triangle revolves.

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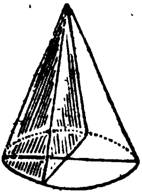
* This is Euclid's definition of a cone, and is that which is generally best understood by a learner; but the following one is more general.

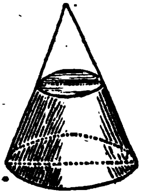
Conceive the right line CB to move upon the fixed point C as a centre, and so as continually to touch the circumference of the circle AB, placed in any position, except in that of a plane which passes through the said point; and then that part of the line which is intercepted between the fixed point and the periphery of the circle will generate the convex superficies of a cone.

4. The *base of a cone* is the circle which is described by the revolving leg of the triangle.

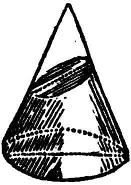
5. If a cone be cut through the vertex, by a plane which also cuts the base, the section will be a *triangle.*



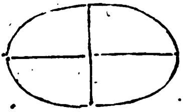
6. If a cone be cut into two parts, by a plane parallel to the base, the section will be a *circle.*



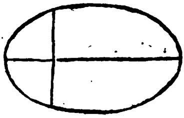
7. If a cone be cut by a plane which passes through its two slant sides in an oblique direction, the section will be an *ellipsis.*



8. The longest straight line that can be drawn in an ellipsis is called the *transverse axis;* and a line drawn perpendicular to the transverse axis, passing through the centre of the ellipse, and terminated both ways by the circumference, is called the *conjugate axis.*

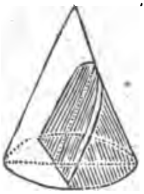


9. An *ordinate* is a right line drawn from any point of the curve, perpendicular to either of the diameters.

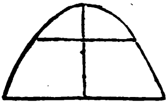


10. An *abscissa* is that part of the diameter which is contained between the vertex and the ordinate.

11. If a cone be cut by a plane, which is parallel to either of its slant sides, the section will be a *parabola*.



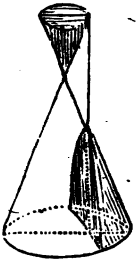
12. The *axis of a parabola* is a right line drawn from the vertex, so as to divide the figure into two equal parts.



13. The *ordinate* is a right line drawn from any point in the curve perpendicular to the axis.

14. The *abscissa* is that part of the axis which is contained between the vertex and the ordinate.

15. *If a cone be cut into two parts, by a plane, which, being continued, would meet the opposite cone, the section is called an *hyperbola*.



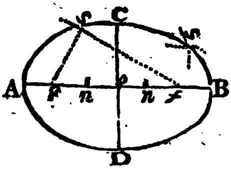
16. The *transverse diameter*, or axis of an hyperbola, is that part of the axis intercepted between the two opposite cones.

17. The *conjugate diameter* is a line drawn through the centre perpendicular to the transverse.

18. An *ordinate* is a line drawn from any point in the curve perpendicular to the axis; and the *abscissa* is the distance intercepted between that ordinate and the vertex.

PROBLEM I.

To describe an ellipsis, the transverse and conjugate diameters being given.



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* The two opposite cones, in this definition, are supposed to be generated together, by the revolution of the same line.

All the figures which can possibly be formed by the cutting of a cone, are mentioned in these definitions, and are the five follow-

*Construction.** 1. Draw the transverse and conjugate diameters, AB, CD, bisecting each other perpendicularly in the centre *o*.

2. With the radius A*o*, and centre C, describe an arc cutting AB in F*f*; and these two points will be the foci of the ellipse.

3. Take any number of points *nn*, &c. in the transverse diameter AB, and with the radii A*n*, *n*B, and centres F*f*, describe arcs intersecting each other in *s*, *s*, &c.

4. Through the points *s*, *s*, &c. draw the curve A*s*CBD, and it will be the circumference of the ellipse required.

PROBLEM II.

In an ellipsis, any three of the four following terms being given, viz. the transverse and conjugate diameters, an ordinate and its abscissa, to find the fourth.

CASE I.

When the transverse, conjugate, and abscissa are known, to find the ordinate.

---

ing ones: *viz.* a *triangle*, a *circle*, an *ellipsis*, a *parabola*, and an *hyperbola;* but the last three only are usually called the *conic sections.*

* It is a known property of the ellipse, that the sum of two lines drawn from the foci, to meet in any point in the curve, is equal to the transverse diameter, and from this the truth of the construction is evident.

From the same principle is also derived the following method of describing an ellipse, by means of a string and two pins.

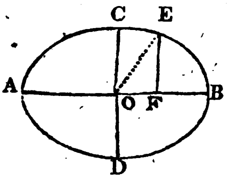
Having found the foci F, *f*, as before, take a thread of the length of the transverse diameter, and fasten its ends with two pins in the points F, *f*; then stretch the thread F*sf* to its greatest extent, and it will reach to the point *s* in the curve; and by moving a pencil round within the thread, keeping it always stretched, it will trace out the curve required.

RULE.*

As the transverse diameter is to the conjugate,
So is the square root of the rectangle of the two abscissas,
To the ordinate which divides them.

EXAMPLES.

1. In the ellipsis ADBC, the transverse diameter AB is 120, the conjugate diameter CD is 40, and the abscissa BF 24: what is the length of the ordinate EF?



Here 120 (AB) : 40 (CD) : : $\sqrt{96\times24}$ (AF × FB) : $\frac{40}{120}\sqrt{96\times24}=\frac{1}{3}\sqrt{2304}=\frac{1}{3}\times48=16=$EF *the ordinate required.*

2. If the transverse diameter be 35, the conjugate 25, and the abscissa 28: what is the ordinate? Ans. 10.

CASE II.

When the transverse, conjugate, and ordinate are known, to find the abscissa.

---

[*] Let t=the transverse diameter, c=conjugate, x=any abscissa, and y=ordinate. Then will the general equation expressing the property of the ellipse, be $t^2 : c^2 :: x\times(t-x) : y^2$; and from this the four rules here given are deduced, the one above being $y=\frac{c}{t}\sqrt{x\times t-x}$.

RULE.*

As the conjugate diameter is to the transverse,

So is the square root of the difference of the squares of the ordinate and semi-conjugate,

To the distance between the ordinate and centre.

And this distance being added to and subtracted from the semi-transverse, will give the two abscissas required.

EXAMPLES.

1. The transverse diameter AB is 120, the conjugate diameter CD is 40, and the ordinate FE is 16; what is the abscissa FB?

Here $40(CD):120(AB)::\sqrt{20^2-16^2}(\sqrt{OB^2-FE^2}):$ $\frac{120}{40}\sqrt{20^2-16^2}=3\sqrt{400-256}=3\sqrt{144}=3\times12=36=$ OF, *the distance from the centre.*

$$\left.\begin{array}{r}\textit{Whence}\ \ 60(OB)-36(OF)=24=BF\\ \textit{And}\ \ 60(OA)+36(OF)=96=AF\end{array}\right\}=\textit{two abscissas}$$
required.

2. What are the two abscissas to the ordinate 10, the diameters being 35 and 25? Ans. 7 *and* 28.

CASE III.

When the conjugate, ordinate, and abscissa are known, to find the transverse.

RULE.†

1. To or from the semi-conjugate, according as the greater or less abscissa is used, add, or subtract the square

---

* This rule in algebraic terms is as follows: The greater abscissa $x=\frac{t}{2}+\frac{t}{c}\sqrt{\tfrac{1}{4}c^2-y^2}$ *or the less* $x=\frac{t}{2}-\frac{t}{c}\sqrt{\tfrac{1}{4}c^2-y^2}$

† This rule in algebraic terms is as follows: $t=(c+2$

root of the difference of the squares of the ordinate and semi-conjugate.

2. Then, as this sum or difference, is to the abscissa, So is the conjugate to the transverse.

EXAMPLES.

1. The conjugate diameter CD is 40, the ordinate EF is 16, and the abscissa FB 24: required the transverse AB.

Here $20-\sqrt{20^2-16^2}(\sqrt{OC^2-EF^2})=20-12=8$.
And 8 : 24 :: 40 :120, *the transverse diameter required.*

2. If an ordinate and its lesser abscissa be 10 and 7, and the conjugate 25, what is the transverse ? Ans. 35.

CASE IV.

The transverse, ordinate, and abscissa being given, to find the conjugate.

RULE.*

As the square root of the product of the two abscissas, is to the ordinate, so is the transverse diameter, to the conjugate.

EXAMPLES.

1. The transverse AB is 120, the ordinate EF 16, and the abscissa FB 24: required the conjugate.

$\sqrt{\frac{1}{4}c^2-y^2})\times\frac{cx}{2y^2}$, or $t=(c-2\sqrt{\frac{1}{4}c^3-y^2})\times\frac{cx}{2y^2}$ according as the greater or less abscissa is used.

* The rule in algebraic terms is $ty\times\frac{1}{\sqrt{tx-xx}}=c$, the conjugate, or shortest diameter.

Here $\sqrt{24\times96}$ ($\sqrt{\mathrm{BF}\times\mathrm{AF}}$) : 16 (EF) : : $\overline{120}$ (AB) : $\overline{16\times120}\div\sqrt{24\times96}=\overline{16\times120}\div\sqrt{2304}=\overline{16\times120}\div 48=\frac{16\times120}{48}=\frac{120}{3}=40$ *the conjugate diameter required.*

2. The transverse diameter is 35, the ordinate is 10, and its abscissa 7: what is the conjugate ? Ans. 25.

PROBLEM III.

To find the circumference of an ellipse, the transverse and conjugate diameters being known.

RULE.*

Multiply the square root of half the sum of the squares of the two diameters by 3.1416, and the product will be the circumference *nearly.*

---

* *Demon.* Let t=transverse diameter, c=conjugate, p=3.1416, and $d=1-\frac{c}{t^2}$. Then will $pt\times(1-\frac{d}{2.2}-\frac{3d^2}{2.2.4.4}-\frac{3.3.5d^3}{2.2.4.4.6.6}$ &c.)= circumference of the ellipse, as is shown by the the writers on fluxions.

Now the rule given above is $p\sqrt{\frac{t^2+c^2}{2}}=pt\sqrt{\frac{1}{2}+\frac{c^2}{2t^2}})$ $=pt\sqrt{(1-\frac{1}{2}+\frac{c^2}{2t^2})}$=(by substitution) $pt\sqrt{(1-\frac{d}{2})}=pt\times(1-\frac{d}{2.2}-\frac{d^2}{2^3.4}-\frac{3d^3}{2^4.4.6})$&c. But the first three terms of this series differ from the first three terms of the former only by $\frac{d^2}{64}$; therefore the rule is shown to be an approximation. Q. E. D.

Rule 2. Multiply ½ the sum of the two diameters by 3.1416, and the product will give the circumference *exact enough to answer most practical purposes.*

Rule 3. Find the circumference both from the last rule and that given above, and ½ the sum of the results will give the circumference *extremely near.*

EXAMPLES.

1. The transverse diameter is 24, and the conjugate 20: required the circumference of the ellipse.

Here $\sqrt{\frac{AB^2+CD^2}{2}} = \sqrt{\frac{24^2+20^2}{2}} = \sqrt{\frac{576+400}{2}} =$ $\sqrt{288+200} = \sqrt{488} = \sqrt{22.09}$.

And $22.09 \times 3.1416 = 69.397944$ *the circumference required.*

2. The axes are 24 and 18: what is the circumference?
Ans. 66.6434.

PROBLEM IV.

To find the area of an ellipse, the transverse and conjugate diameters being given.

RULE.*

Multiply the transverse diameter by the conjugate, and the product again by .7854, and the result will be the area.

Or multiply .7854 by one axe, and the product by the other.

Note.—If a = semi-transverse BO, C = semi-conjugate CO, and x = distance OF, of the ordinate EF from the centre, then will the arc CE be $= x \times (1 + \frac{c^2}{6a^4}x^2 + \frac{4a^2c^2 - c^4}{40a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^{12}}x^6$ &c.

The following may serve as a practical rule for finding the length of the arc CE.

Find the length of a circular arc intercepted by OE and OC, and whose radius is ½ the sum of OE and OC, and it will be the elliptic arc CE *nearly*.

* The demonstration of this rule is contained in that of the next problem.

EXAMPLES.

1. What is the area of an ellipse whose transverse diameter is 24, and the conjugate 18?

Here $24 \times 18 \times .7854 = 339.2928 =$ *area required.*

2. If the axes of an ellipse be 35 and 25, what is the area? Ans. 687.225.

3. Required the area of an ellipsis whose two axes are 70 and 50. Ans. 2748.9.

PROBLEM V.

To find the area of an elliptic segment, whose base is parallel to either of the axes of the ellipse.

RULE.*

1. Divide the height of the segment by that axe of the ellipse of which it is a part, and find in the table a circular segment, whose versed sine is equal to the quotient.

---

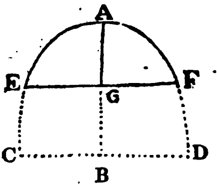
* *Demon.* Let the transverse diameter $2\text{AB} = t$, the conjugate $\text{CD} = c$, $\text{AG} = x$, and $\text{EG} = y$; then by the property of the curve we shall have $y = \frac{c}{t}\sqrt{tx - x^2}$, and the fluxion of the area $\text{EAF} = (y\dot{x}) = \frac{c}{t} \times \dot{x}\sqrt{tx - x^2}$. But $\dot{x} \times \sqrt{tx - x^2}$ is known to express the fluxion of the corresponding circular segment, whose versed sine is x, and the diameter t. Let the fluent of this expression therefore be denoted by A, and then the fluent of $\frac{c}{t} \times \dot{x}\sqrt{tx - x^2}$ will be $= \frac{c}{t} \times \text{A}$, from whence the rule is derived. Q. E. I.

Corol. The ellipse is equal to a circle whose diameter is a mean proportional between the two axes, and from hence the rule is formed for the whole ellipse,

2. Multiply the segment thus found, and the two axes of the ellipse continually together, and the product will give the area required.

EXAMPLES.

1. Required the area of the elliptic segment EAF, whose height AG is 10, and the axis of the ellipse 2AB and CD, 35 and 25 *respestively*.



Here $\frac{10.0000}{35}=\frac{2.0000}{7}=.2857=$ *tabular versed sine.*

And the tabular segment belonging to this is .185153.

Whence .185153 × 35 (2AB) × 25(CD) = 6.480355 × 25 = 162.0088 = *area of the segment required.*

2. What is the area of an elliptic segment cut off by a double ordinate parallel to the conjugate axe, at the distance of 36 from the centre, the axes being 120 and 40 ?

Ans. 536.7504.

3. What is the area of a segment cut off by an ordinate parallel to the transverse diameter, whose height is 5, the axes being 35 and 25 ? Ans. 97.845125.

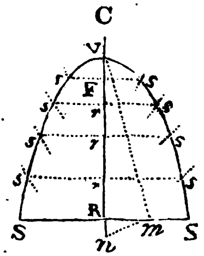
The area of an elliptic segment may also be found by the following rule.

Find the corresponding segment of the circle described upon the same axe to which the base of the segment is perpendicular.

Then as this axe is to the other axe, so is the circular segment to the elliptic segment.

PROBLEM VI.

To describe a parabola, any ordinate to the axe and its abscissa being given.



**Construction.* 1. Bisect the given ordinate RS in *m;* join V*m*, and draw *mn* perpendicular to it, meeting the axe in *n*.

2. Make VC and VF each equal to R*n*, and F will be the focus of the curve.

3. Take any number of points *r*, *r*, &c. in the axe, through which draw the double ordinates S*r*S, &c. of an indefinite length.

---

* Since V*mn* is a right-angled triangle, and *m*R is perpendicular to V*n*, $\text{V} \times \text{RR}n = \text{VR} \times \text{VF} = \text{R}m^2 = \frac{1}{4}\text{RS}^2$, which is a known property of the parabola when F is the focus. And, because $s\text{F}^2 = cr^2 - r\text{F}^2 = \overline{cr + r\text{F}} \times \overline{cr - r\text{F}} = \overline{cr + r\text{F}} \times c\text{F} = 2\text{V}r \times 2\text{VF} = \text{V}r \times 4\text{VF}$, therefore, *s* is a point in the curve of a parabola, and the same may be shown of any other point *s*.

Besides the methods above, for finding the focus, it may be found arithmetically as follows:

Divide the square of the ordinate by 4 times the abscissa, and the quotient will be the focal distance VF.

Several other methods of doing this, as well as of describing the curve itself, may also be found in Emerson's Conic Sections, and other performances.

4. With the radii CF, C*r*, &c. and centre F, describe arcs cutting the corresponding ordinates in the points *s*, *s*, &c. and the curve SVS drawn through all the points of intersection will be the parabola required.

Note.—The line *s*F*s* passing through the focus F is called the parameter.

PROBLEM VII.

In a parabola, any three of the four following terms being given, viz. any two ordinates and their two abscissas, to find the fourth.

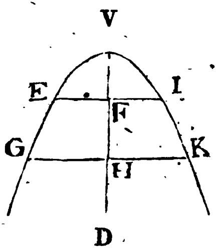
RULE.*

As any abscissa is to the square of its ordinate, so is any other abscissa to the square of its ordinate.

Or as the square root of any abscissa is to its ordinate, so is the square root of any other abscissa to its ordinate.

EXAMPLES.

1. The abscissa VF is 9, and its ordinate EF 6; required the ordinate GH, whose abscissa VH is 16.



Here $\sqrt{9}(\sqrt{VF}) : 6(EF) :: \sqrt{16}(\sqrt{VH}) : \frac{6\times\sqrt{16}}{\sqrt{9}} = \frac{6\times 4}{3} = \frac{24}{3} = 8 =$ *ordinate* GH.

---

* If x and X be any two abscissas, and y and Y their corresponding ordinates, the equation of the curve will be $xY^2 = Xy^2$, which is the same as the rule.

Or,

$9(VF) : 36(EF^2) :: 16(VH) : \frac{16\times36}{9} = 16\times4 = 64 =$ GH^2, *or* $8 = GH$ *as before.*

2. The two abscissas are 9 and 16, and their corresponding ordinates 6 and 8; from any three of these to find the fourth.

PROBLEM VIII.

To find the length of any arc of a parabola, cut off by a double ordinate.

RULE.*

To the square of the ordinate add $\frac{4}{3}$ of the square of the abscissa, and twice the square root of the sum will be the length of the curve required.

---

* *Demon.* Let x=any abscissa, y=its ordinate, a=$\frac{1}{2}$ the parameter of the axe, and $q=\frac{y}{a}$. Then it is shown by the writers on fluxions, that, $aq\sqrt{1+q^2}+a\times$ hyp. log. of $(q+\sqrt{1+q})=2y\times(1+\frac{q^2}{2.3}-\frac{q^4}{2.4.5}+\frac{3q^6}{2.4.6.7}-\frac{3.5q^8}{2.4.6.8.9}$ &c.)$=c$=length of the curve. But $\sqrt{1+\frac{1}{3}q^2}=1+\frac{q^2}{2.3}-\frac{q^4}{2.4.9}+\frac{3q^6}{2.4.6.27}$ &c. agreeing with the former in the two first terms.

Therefore $\frac{c}{2y}=\sqrt{1+\frac{1}{3}q^2}$ *nearly*; and consequently $c=2y\sqrt{1+\frac{1}{3}q^2}=2\sqrt{y^2+\frac{4}{3}x^2}$ the same as the rule. Q. E. D.

Note.—This rule must be used only when the abscissa does not exceed half the ordinate. The length of the curve in other cases must be found by means of hyperbolic logarithms, as is shown by writers on fluxions.

EXAMPLES.

1. The abscissa VH is 2, and its ordinate GH 6; what is the length of the arc GVK?

Here $2^2(VH^2)\times\frac{4}{3}+36(GH^2)=\frac{4\times4}{3}+36=\frac{16}{3}+36=$ 5.333, &c.$+36=41.333333$.

```
And 41.333333(6.429
    36            2
    ——          —————
124)533        12.858=length of the arc.
    496
    ———
1282)3733
     2564
     ————
12849)116933
      115641
      ——————
        1292
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PROBLEM IX.

To find the area of a parabola, its base and height being given.

RULE.*

Multiply the base by the height, and $\frac{2}{3}$ of the product will be the area required.

---

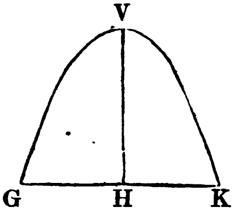
* *Demon.* Let VH$=x$, GH$=y$, and the parameter$=p$. Then $px=y^2$, or $\sqrt{px}=y$ by the nature of the curve. Whence the fluxion of the area $(=y\dot{x})=\dot{x}\sqrt{px}$ and its fluent$=\frac{2}{3}x\times\sqrt{px}$.

But because $y=\sqrt{px}$, therefore $\frac{2}{3}x\times\sqrt{px}=\frac{2}{3}xy=$area of the parabola, which is the same as the rule.

Coroll. Every parabola is $=\frac{2}{3}$ of its circumscribing parallelogram.

EXAMPLES.

1. What is the area of the parabola GVK, whose height VH is 12, and the base or double ordinate GK 16?



Here $16(\mathrm{GK}) \times 12(\mathrm{VH}) \times \frac{2}{3} = \frac{16 \times 12 \times 2}{3} = 16 \times 4 \times 2 =$ $128 =$ *area required.*

1. The abscissa is 12, and the double ordinate or base 38; what is the area? Ans. 304.

3. What is the area of a parabola whose abscissa is 10, and ordinate 8? Ans. $106\frac{2}{3}$.

PROBLEM X.

To find the area of a frustum of a parabola.

RULE.*

Divide the difference of the cubes of the two ends of the frustum by the difference of their squares, and this quotient multiplied by $\frac{2}{3}$ of the altitude, will give the area required.

---

* *Demon.* Let $\mathrm{D}=\mathrm{GK}$, $d=\mathrm{EI}$, and $a=\mathrm{FH}$.

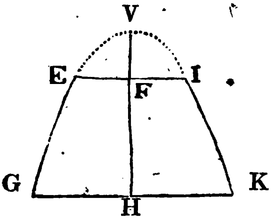
Then by the nature of the curve $\mathrm{D}^2-d^2 : a :: \mathrm{D}^2 : \frac{\mathrm{AD}^2}{\mathrm{D}^2-d^2}=\mathrm{VH}$, and $\mathrm{D}^2-d^2 : a :: d^2 : \frac{ad^2}{\mathrm{D}^2-d^2}=\mathrm{VF}$.

And therefore $\frac{2}{3} \times \frac{a\mathrm{D}^2}{\mathrm{D}^2-d^2} - \frac{2}{3} \times \frac{ad^2}{\mathrm{D}^2-d^2} = \frac{2}{3}a \times \frac{\mathrm{D}^3-d^3}{\mathrm{D}^2-d^2}$ $=$ area of the frustum. Q. E. D.

Note.—Any parabolic frustum is equal to a parabola of the same altitude, whose base is equal to one end of the frustum, increased

EXAMPLES.

1. In the parabolic frustum GEIK, the two parallel ends EI and GK are 6 and 10, and the altitude, or part of the abscissa FH, is 3 ; what is the area ?



Here $\overline{10^3-6^3}(GK^3-EI^3) \div \overline{10^2-6^2}(GK^2-EI^2) = \frac{10^3-6^3}{10^2-6^2} = \frac{1000-216}{100-36} = \frac{784}{64} = \frac{98}{8} = \frac{49}{4} = 12.25.$

And $12.25 \times \frac{2 \times 3}{3} = 12.25 \times 2 = 24.5 =$ *area required.*

2. The greater end of the frustum is 24, the lesser end is 20, and their distance $5\frac{1}{2}$; what is the area ?

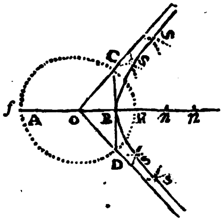
Ans. 121.3333.

3. Required the area of the parabolic frustum, the greater end of which is 10, the less 6, and the height 4.

Ans. $32\frac{2}{3}$.

PROBLEM XI.

To construct an hyperbola, the transverse and conjugate diameters being given.



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by a third proportional to the sum of the two ends, and the other end.

[*] *Construction.* 1. Make AB the transverse diameter, and CD perpendicular to it, the conjugate.

2. Bisect AB in O, and from O with the radius OC, or OD, describe the circle D*f*CF, cutting AB produced in F and *f*, which points will be the two foci.

3. In AB produced take any number of points, *n*, *n*, *&c.* and from F and *f*, as centres, with the distances B*n*, A*n*, as radii, describe arcs cutting each other in *s*, *s*, *&c.*

4. Through the several points *s*, *s*, *&c.* draw the curve *s*B*s*, and it will be the hyperbola required.

5. If straight lines be drawn from the point O, through the extremities CD of the conjugate axis, they will be the asymptotes of the hyperbola, whose property it is to approach continually to the curve, and yet never to meet it.

PROBLEM XII.

In an hyperbola, any three of the four following terms being given, viz. the transverse and conjugate diameters, an ordinate, and its abscissa, to find the fourth.

CASE I.

The transverse and conjugate diameters, and the two abscissas being known, to find the ordinate.

RULE.[†]

As the transverse diameter,
Is to the conjugate;
So is the square root of the product of the two abscissas,
To the ordinate required.

---

[*] The sum of two lines drawn from the foci of an ellipse to any point in the curve, is equal to its transverse diameter.

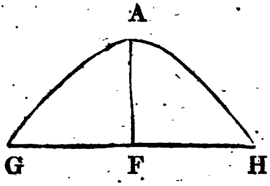
In like manner the difference of two lines drawn from the foci of any hyperbola to any point in the curve, is equal to its transverse diameter, as is shown by the writers on *conics.*

But the arcs intersecting each other in *s*, *s*, &c. are described from the foci *f* and F, and with the distances A*n*, and B*n*, whose difference is AB, and therefore the points *s*, *s*, &c. are in the curve of an hyperbola.

[†] Let t=transverse diameter, c=conjugate, x=abscissa, and

EXAMPLES.

1. In the hyperbola GAH, the transverse diameter is 120, the conjugate 72, and the abscissa AF is 40; required the ordinate FH.



$$120\ (\textit{trans.}) : 72\ (\textit{conj.}) :: \surd(160 \times 40) : \frac{72\times\surd(160\times40)}{120} = \frac{6\times\surd(160\times40)}{10} = \tfrac{3}{5}\surd(160\times40) = \tfrac{3}{5}$$

$$\surd 6400 = \tfrac{3}{5}\times80 = \frac{3\times80}{5} = 3\times16 = 48 = \textit{ordinate}\ \text{FH}.$$

2. The transverse diameter is 24, the conjugate 21, and the less abscissa 8; what is its ordinate?

Ans. 14.

3. The transverse diameter of an hyperbola is 50, the conjugate 30, and the less abscissa 12; required the ordinate. Ans. 16.3658.

CASE II.

The transverse and conjugate diameters and an ordinate being given, to find the two abscissas.

RULE.

As the conjugate diameter is to the transverse,

So is the square root of the sum of the squares of the ordinate and semi-conjugate,

To the distance between the ordinate and the centre, or half the sum of the abscissas.

---

y=ordinate. Then the general property of the curve is $t^2 : c^2 :: x\times(t+x) : y^2$; and from this analogy all the cases of this problem are deduced.

Note.—In the hyperbola, the less abscissa added to the axis, gives the greater.

Then the sum of this distance and the semi-transverse will give the greater abscissa, and their difference the less abscissa required.*

EXAMPLES.

The transverse diameter is 120, the conjugate 72, and the ordinate 48; what are the two abscissas?

1296 = *square of the semi-conjugate.*
2304 = *square of the ordinate.*

3600(60 = *square root.*
36

00

As 72 : 120 :: 60
120

72)7200(100 = ½ *sum of the abscissas.*
72 60 = *semi-transverse.*

160 = *greater abscissa.*

40 = *less abscissa.*

2. The transverse and conjugate diameters are 24 and 21; required the two abscissas to the ordinate 14.

Ans. 32 *and* 8.

3. The transverse being 60, and the conjugate 36; required the two abscissas to the ordinate 24.

Ans. 80 *and* 20.

---

* This rule in species is $\frac{t}{c}\sqrt{\frac{1}{4}c^2+y^2}\pm\frac{1}{2}t=x$, = greater or less abscissa, according as the upper or under sign is used.

CASE III.

The transverse diameter, the two abscissas and the ordinate being given, to find the conjugate.

RULE.

As the square root of the product of the two abscissas,
Is to the ordinate;
So is the transverse diameter,
To the conjugate.*

EXAMPLES.

1. The transverse diameter is 120, the ordinate is 48, and the two abscissas are 160 and 40; required the conjugate.

$$\begin{array}{r} 160 \\ 40 \\ \hline \dot{6}4\dot{0}0(80 \\ 64 \\ \hline 00 \end{array}$$

As 80 : 48 : : 120 *the transverse axis.*

$$\begin{array}{r} 48 \\ \hline 960 \\ 480 \\ \hline 80)5760 \\ \hline 72 \end{array}$$

72=*conjugate required.*

---

* This rule, expressed algebraically, is $ty \div \sqrt{x \times (t \times x)} = c =$ conjugate diameter.

2. The transverse diameter is 24, the ordinate 14, and the abscissas 8 and 32; required the conjugate.

Ans. 21.

CASE IV.

The conjugate diameter, the ordinate and two abscissas being given, to find the transverse.

RULE.

1. Add the square of the ordinate to the square of the semi-conjugate, and find the square root of their sum.

2. Take the sum or difference of the semi-conjugate and this root, according as the less or greater abscissa is used, and then say,

As the square of the ordinate,
Is to the product of the abscissa and conjugate;
So is the sum, or difference, above found,
To the transverse required.*

EXAMPLES.

1. The conjugate diameter is 72, the ordinate is 48, and the less abscissa 40; what is the transverse?

Here $\sqrt{48^2+36^2}=\sqrt{2304+1296}=\sqrt{3600}=60$:

And $60+36=96$.

Also $72\times 40=2880$ =*product of the abscissa and conjugate. Whence,*

---

* This rule in algebraic terms is $\frac{cx}{y^2}\times(\sqrt{\frac{1}{4}c^2+y^2}\pm\frac{1}{2}c)$ $=t=$transverse diameter.

As 2304 : 2880 : : 96

```
         96
       -----
       17280
      25920
      -----
2304)276480(120
      2304
      -----
       4608
       4608
      -----
```

120=*transverse required.*

2. The conjugate diameter is 21, the less abscissa 8, and its ordinate 14; required the transverse. Ans. 24.

3. Required the transverse diameter of the hyperbola, whose conjugate is 36, the less abscissa being 20, and ordinate 24. Ans. 60.

PROBLEM XIII.

To find the length of any arc of an hyperbola, beginning at the vertex.

RULE.*

1. As the transverse is to the conjugate, so is the conjugate to the parameter.

---

* *Demon.* Let t=semi-transverse axe, c=semi-conjugate, x=ordinate, and y=abscissa. Then will $y\times(1+\frac{t^2}{6c^4}y-\frac{t^4+4t^2c^2}{40c^8}y^4+\frac{t^6+4t^4c^2+8tc^4}{112c^{12}}y^6$, &c.)=length of the arc, as is shown by the writers on fluxions.

But $x=\frac{t}{c}\sqrt{c^2+y^2}-a$, and $\frac{2c}{t}$=parameter=p, by the

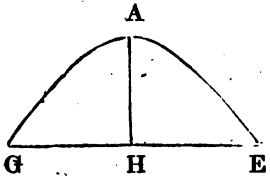
2. To 19 times the transverse, add 21 times the parameter of the axis, and multiply the sum by the quotient of the abscissa divided by the transverse.

3. To 9 times the transverse, add 21 times the parameter, and multiply the sum by the quotient of the abscissa divided by the transverse.

4. To each of the products, thus found, add 15 times the parameter, and divide the former by the latter; then this quotient being multiplied by the ordinate will give the length of the arc *nearly.*

EXAMPLES.

1. In the hyperbola GAE, the transverse diameter is 80, the conjugate 60, the ordinate GH 10, and the abscissa AH 2.1637; required the length of the arc AG.



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nature of the curve. Consequently the rule becomes $(15p+\overline{19t+21p}\times\frac{x}{t})\div(15p+\overline{9t+21p}\times\frac{x}{t})\times y=(15pt+19tx+21px)\div(15pt+9tx+21px)\times y=y\times:1+\frac{2x}{3p}-\frac{2x^2}{5p}$ &c. which by substituting the values of x and p, and expanding the terms, gives a series, agreeing nearly in the first three terms with the former; and therefore, the rule is an approximation.

If t=semi-transverse, c=semi-conjugate, and y=ordinate drawn from the end of the required arc; then $y\times(1+\frac{t^2y^2}{6c^4}\text{A}-\frac{t^2+4c^2}{c^2}.\frac{3y^2}{20}\text{B}+\frac{t^4+4t^2c^2+8c^4}{t^2+4c^2}.\frac{5y^2}{14c^2}\text{C}$, &c.$)=$ length of the arc.

Here $80 : 60 :: 60 : \frac{60\times60}{80}=\frac{3\times60}{4}=3\times15=45=$ *parameter.*

And $(\overline{80\times19}\times\overline{45\times21})\times\frac{2.1637}{80}=\overline{1520+945}\times.02704$ $=2465\times.02704=66.6536.$

Also $(\overline{80\times9}+\overline{45\times21})\times\frac{2.1637}{80}=\overline{720+945}\times.02704=$ $1665\times.02704=45.0216.$

Whence $\overline{675+66.6536}\div\overline{675+45.0216}=741.6536\div$ $720.0216=1.03004$; *and* $1.03004\times10=10.3004=$ *length of the arc required.*

2. The transverse diameter of an hyperbola is 120, the conjugate 72, the ordinate 48, and the abscissa 40: required the length of the curve. Ans. 62.6496.

3. Required the whole length of the curve of an hyperbola, to the ordinate 10; the transverse and conjugate axes being 80 and 60. Ans. 20.6006.

PROBLEM XIV.

To find the area of an hyperbola, the transverse, conjugate and abscissa being given.

RULE.*

1. To the product of the transverse and abscissa, add $\frac{5}{7}$ of the square of the abscissa, and multiply the square root of the sum by 21.

---

* *Demon.* Let $t=$ transverse diameter, $c=$ conjugate, $x=$ abscissa, $y=$ ordinate, and $z=\frac{x}{t^2+x}$. Then it is well known that $4xy\times(\frac{1}{3}-\frac{1}{1.3.5.}z-\frac{1}{3.5.7.}z^2-\frac{1}{5.7.9.}z^3,$ &c.$)=$ area of the hyperbola.

2. Add 4 times the square root of the product of the transverse and abscissa, to the product last found, and divide the sum by 75.

3. Divide 4 times the product of the conjugate and abscissa by the transverse, and this last quotient multiplied by the former will give the area required *nearly.*

EXAMPLES.

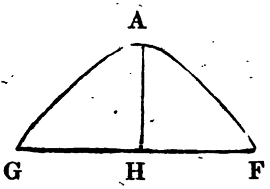
In the hyperbola GAF, the transverse axis is 30, the conjugate 18, and the abscissa or height AH is 10; what is the area?

But $\frac{ty}{\sqrt{ta+x^2}} = c =$ conjugate axis, by the nature of the hyperbola. Consequently the expression for the rule $= \frac{4xc}{t} \times \frac{21\sqrt{tx+\frac{5}{7}x}+4\sqrt{tx}}{75} = 4xy \times \frac{21\sqrt{tx+\frac{5}{7}x^2}+4\sqrt{tx}}{\sqrt{tx-x^2}}$ And this thrown into a series will very nearly agree with the former; which shows the rule to be an approximation.

Q. E. I.

Rule 2. If $2Y$, $2y =$ bases, V, and v their distances from the centre, and the other letters as before, then will $VY - vy - \frac{tc}{4} \times$ hyp. log. of $\frac{aY+cV}{ty+cv} =$ area of the frustum of the hyperbola.

Rule 3. If t be put = transverse axis, c = conjugate, and x = abscissa, the area of a segment of an hyperbola, cut off by a double ordinate, will be $= \frac{4\sqrt{tx+\frac{3}{4}x^2}+\sqrt{tx}}{15} \times \frac{4cx}{t}$ very nearly.



Here $21\sqrt{(30\times10+\frac{5}{7}\times10^2)}=21\sqrt{300+500\div7}=21\sqrt{300+71.42857}=21\sqrt{371.42857}=21\times19.272=404.712.$

And $(4\sqrt{30\times10}+404.712)\div75=(4\sqrt{300}+404.712)\div75=(4\times17.3205+404.712)\div75=(69.282+404.712)\div75=473.994\div75=6.3199.$

Whence $\frac{18\times10\times4}{30}\times6.3199=\frac{18\times4}{3}\times6.3199=6\times4\times6.3199=24\times6.3199=151.6776=$ *area required.*

2. The transverse diameter is 100, the conjugate 60, and the less abscissa 50; what is the area of the hyperbola?
Ans. 3220.363472.

3. Required the area of the hyperbola to the abscissa 25, the two axes being 50 and 30. Ans. 805.0909.

OF THE

MENSURATION

OF

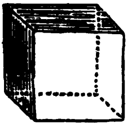
SOLIDS.

DEFINITIONS.

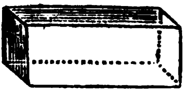
1. The *measure* of any solid body, is the whole capacity or content of that body, when considered under the triple dimensions of length, breadth, and thickness.

2. A *cube* whose side is one inch, one foot, or one yard, &c. is called the *measuring unit*; and the content or solidity of any figure is computed by the number of those cubes contained in that figure.

3. A *cube* is a solid contained by six equal square sides.

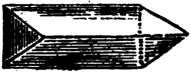


4. A *parallelopipedon* is a solid contained by six quadrilateral planes, every opposite two of which are equal and parallel.

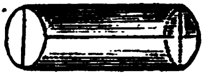


5. A *prism* is a solid whose ends are two equal, parallel, and similar plane figures, and whose sides are parallelograms.

Note.—When the ends are triangles it is called a *triangular prism;* when they are squares, a *square prism;* when they are pentagons, a *pentagonal prism,* &c.

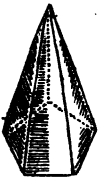


6. A *cylinder* is a solid described by the revolution of a right angled parallelogram about one of its sides, which remains fixed.



7, A * *pyramid* is a solid whose sides are all triangles meeting in a point at the vertex, and the base any plane figure whatever.

Note.—When the base is a triangle, it is called a *triangular pyramid;* when a square, it is called a *square* or *quadrangular pyramid;* when a pentagon, it is called a *pentagonal pyramid,* &c.



8. A *sphere* is a solid described by the revolution of a semicircle about its diameter, which remains fixed.

9. The centre of a sphere is a point within the figure, every where equally distant from the convex surface of it.

10. The diameter of the sphere is a straight line passing

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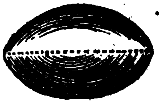
* The definition of a cone has been given already.

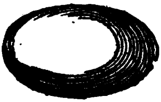
through the centre, and terminated both ways by the convex superficies.

11. A *circular spindle* is a solid generated by the revolution of a segment of a circle about its chord, which remains fixed.



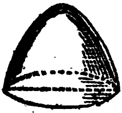
12. A *spheroid* is a solid generated by the revolution of a semi-ellipsis about one of its diameters, which is considered as quiescent.

The spheroid is called *prolate*, when the revolution is made about the transverse diameter, and *oblate* when it is made about the conjugate diameter.



13. *Elliptic, parabolic*, and *hyperbolic spindles*, are generated in the same manner as the circular spindle, the double ordinate of the section being always fixed or quiescent.

14. *Parabolic* and *hyperbolic conoids*, are solids formed by the revolution of a semi-parabola or semi-hyperbola about its transverse axis, which is considered as quiescent.



15. The *segment* of a pyramid, sphere, or of any other solid, is a part cut off from the top by a plane parallel to the base of that solid.

16. A *frustum* or *trunk*, is the part that remains at the bottom, after the segment is cut off.

17. The *zone of a sphere*, is that part which is inter-

cepted between two parallel planes; and when those planes are equally distant from the centre, it is called the middle zone of the sphere.

18. The height of a solid is a perpendicular, drawn from its vertex to the base or plane on which it is supposed to stand.

PROBLEM I.

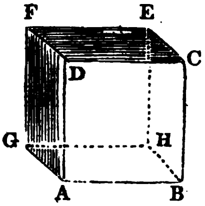
To find the solidity of a cube, the height of one of its sides being given.

RULE.*

Multiply the side of the cube by itself, and that product again by the side, and it will give the solidity required.

EXAMPLES.

1. The side AB, or BC, of the cube ABCDFGHE, is 25.5: what is the solidity ?



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* *Demon.* Conceive the base of the cube to be divided into a number of little squares, each equal to the *superficial measuring unit.*

Then will those squares be the bases of a like number of small cubes, which are each equal to the *solid measuring unit.*

But the number of little squares contained in the base of the cube are equal to the square of the side of that base, as has been shown already.

And consequently, the number of small cubes contained in the whole figure, must be equal to the square of the side of the base multiplied by the height of that figure; or, which is the same thing, *Here* $AB^3 = \overline{25.5}|^3 = 25.5 \times 25.5 \times 25.5 = 25.5 \times 650.25 = 16581.375$ *the answer, or content of the cube.*

2. The side of a cube is 15 inches: what is the solidity?
Ans. 1*ft.* 11*in.* 5*pa.*

3. What is the solidity of a cube whose side is 17.5 inches? *Ans.* 3.1015 *feet.*

PROBLEM II.

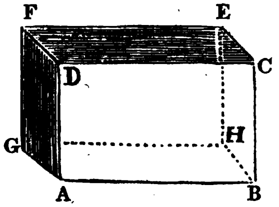
To find the solidity of a parallelopipedon.

RULE.*

Multiply the length by the breadth, and that product again by the depth or altitude, and it will give the solidity required.

EXAMPLES.

1. Required the solidity of a parallelopipedon ABCD FEHG, whose length AB is 8 feet, its breadth FD 4½ feet, and the depth or altitude AD 6¾ feet?



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the square of the side of the base multiplied by the base, is equal to the solidity. Q. E. D.

Note.—The surface of the cube is equal to six times the square of its side.

* The reason of this rule, as well as the following ones for the prism and cylinder, is the same as that for the cube.

Note.—The surface of the parallelopipedon is equal to the sum of the areas of each of its sides or ends.

Here AB×AD×FD=8 × 6.75 × 4.5=54 × 4.5=243 *solid feet, the contents of the parallelopipedon required.*

2. The length of a parallelopipedon is 15 feet, and each side of its square base 21 inches: what is the solidity?

Ans. 45.9375 feet.

3. What is the solidity of a block of marble, whose length is 10 feet, its breadth $5\frac{3}{4}$ feet, and the depth $3\frac{1}{2}$ feet?

Ans. 201.25 feet.

PROBLEM III.

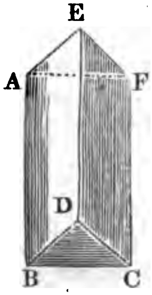
To find the solidity of a prism

RULE.*

Multiply the area of the base into the perpendicular height of the prism, and the product will be the solidity.

EXAMPLES.

1. What is the solidity of the triangular prism ABCFED, whose length AB is 10 feet, and either of the equal sides, BC, CD, or DB, of one of its equilateral ends BCD, $2\frac{1}{2}$ feet?



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* The surface of a prism is equal to the sum of the areas of the two ends and each of its sides.

* *Here* $\frac{1}{4}\times 2.5^2\times\sqrt{3}=\frac{1}{4}\times 6.25\times\sqrt{3}=1.5625\times\sqrt{3}=1.5625\times 1.732=2.70625=$ *area of the base* BCD.

Or, $\frac{2.5+2.5+2.5}{2}=\frac{7.5}{2}=3.75=\frac{1}{2}$ *sum of the sides*, BC, CD, DB, *of the triangle* CDB.

And $3.75-2.5=1.25$, $\therefore$ 1.25, 1.25 *and* $1.25=3$ *differences.*

Whence $\sqrt{3.75\times 1.25\times 1.25\times 1.25}=\sqrt{3.75\times 1.25^3}=\sqrt{7.32421875}=2.7063=$ *area of the base as before,*

And $2.7063\times 10=27.063$ *solid feet, the content of the prism required.*

2. What is the solidity of a triangular prism, whose length is 18 feet, and one side of the equilateral end $1\frac{1}{2}$ feet? Ans. 17.5370265 feet.

3. Required the solidity of a prism whose base is a hexagon, supposing each of the equal sides to be 1 foot 4 inches, and the length of the prism 15 feet. Ans. 69.282ft.

PROBLEM IV.

To find the convex surface of a cylinder.

RULE.†

Multiply the periphery or circumference of the base, by the height of the cylinder, and the product will be the convex surface required.

EXAMPLES.

1. What is the convex surface of the right cylinder ABCD, whose length BC is 20 feet, and the diameter of its base AB 2 feet?

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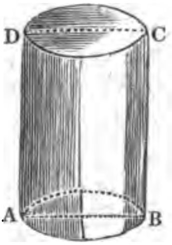
* *See Notes to Prob.* III. *Cor.* 2. *p.* 56.

† *Demon.* If the periphery of the base be conceived to move in a direction parallel to itself, it will generate the convex superficies of the cylinder; and, therefore, the said periphery being multiplied by the length of the cylinder, will be equal to that superficies.

Q. E. D.

Note. If twice the area of either of the ends be added to the convex surface, it will give the whole surface of the cylinder.



Here 3.1416 × 2=6.2832=*periphery of the base* AB.

And 6.2832 × 20=125.6640 *square feet, the convexity required.*

2. What is the convex surface of a right cylinder, the diameter of whose base is 30 inches, and the length 60 inches? *Ans.* 5654.88 *inches.*

3. Required the convex superficies of a right cylinder, whose circumference is 8 feet 4 inches, and its length 14 feet. *Ans.* 116.666, *&c. feet.*

PROBLEM V.

To find the solidity of a cylinder.

RULE:*

Multiply the area of the base by the perpendicular height of the cylinder, and the product will be the solidity.

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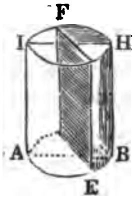
* The four following cases contain all the rules for finding the superficies and solidities of *cylindric ungulas.*

1. *When the section is parallel to the axis of the cylinder.*



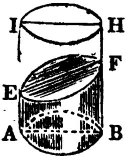
EXAMPLES.

1. What is the solidity of the cylinder ABCD, the diameter of whose base AB is 30 inches, and the height BC 50 inches?

Rule 1. Multiply the length of the arc line of the base by the height of the cylinder, and the product will be the *curve surface.*

2. Multiply the area of the base by the height of the cylinder, and the product will be the *solidity.*

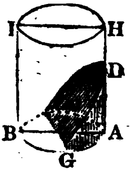
II. *When the section passes obliquely through the opposite sides of the cylinder.*



Rule 1. Multiply the circumference of the base of the cylinder by half the sum of the greatest and least lengths of the ungula, and the product will be the *curve surface.*

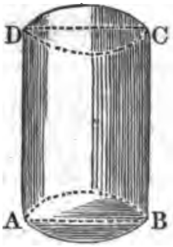
2. Multiply the area of the base of the cylinder by half the sum of the greatest and least lengths of the ungula, and the product will be the *solidity.*

III. *When the section passes through the base of the cylinder, and one of its sides.*



Rule 1. Multiply the sine of half the arc of the base by the diameter of the cylinder, and from this product subtract the product of the arc and cosine.

2. Multiply the difference thus found, by the quotient of the height divided by the versed sine, and the product will be the *curve surface.*



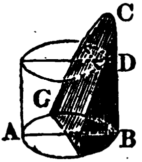
* *Here* $.7854 \times 30^2 = .7854 \times 900 = 706.86 =$ *area of the base* AB.

---

3. From $\frac{2}{3}$ of the cube of the right sine of half the arc of the base, subtract the product of the area of the base and the cosine of the said half arc.

4. Multiply the difference, thus found, by the quotient arising from the height divided by the versed sine, and the product will be the *solidity*.

IV. *When the section passes obliquely through both ends of the cylinder.*



Rule 1. Conceive the section to be continued, till it meets the side of the cylinder produced; then say, as the difference of the versed sines of half the arcs of the two ends of the ungula, is to the versed sine of half the arc of the less end, so is the height of the cylinder to the part of the side produced.

2. Find the surface of each of the ungulas, thus formed, by Case III, and their difference will be the *surface required.*

3. In like manner find the solidities of each of the ungulas, and their difference will be the *solidity required.*

* In working the examples in this and the following rules, .7854 is used for the area, and 3.1416, the circumference of a circle whose diameter is 1; where greater accuracy is required, .7853981634 may be used for the area, and 3.14159265359 for the circumference. *See Note to Prob. IX. Superficies.*

And 706.86 × 50 = 35343 *cubic inches; or* $\frac{35343}{1728} = 20.4531$ *solid feet, the answer required.*

2. What is the solidity of a cylinder whose height is 5 feet, and the diameter of the end 2 feet? Ans. 15.708 feet.

3. What is the solidity of a cylinder whose height is 20 feet, and the circumference of its base 20 feet also? Ans. 636.64 feet.

4. The circumference of the base of an oblique cylinder is 20 feet, and the perpendicular height 19.318; what is the solidity? Ans. 614.93 feet.

PROBLEM VI.

To find the convex surface of a right cone.

RULE.*

Multiply the circumference of the base by the slant height, or the length of the side of the cone, and half the product will be the surface required.

---

* *Demon.* Let AB $=a$, BC $=b$, 3.1416 $=p$, and ED $=y$.

Then $a : b :: y : \frac{by}{a} =$ DC; and $py =$ circumference of the circle ED.

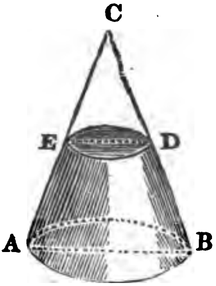
But $py \times \frac{\dot{b}y}{a} =$ fluxion of the surface of CED, and its fluent $= \frac{phy^2}{2a}$ which, when $y=a$, becomes $\frac{pha}{2} =$ convex surface of the whole cone. Q. E. D.

To find the surface of a right pyramid.

Rule. Multiply the perimeter of the base by the length of the side, or slant height of the cone, and half the product will be the surface required.

EXAMPLES.

1. The diameter of the base AB is 3 feet, and the slant height AC or BC 15 feet; required the convex surface of the cone ACB.



Here $3.1416\times3=9.4248=$ *circumference of the base* AB.

Here $\frac{(22.5+15.75)\times26}{2}=\overline{22.5+15.75}\times13=38.25\times 13=497.25=$ *convex surface required.*

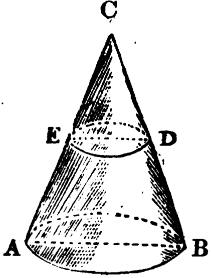
Then P : p : : b (BC) : CD; and, by division, P—p : p : : b—CD(h) : CD$=\frac{ph}{P-p}$; but $p\times(h+\frac{ph}{P-p})$=twice the convex surface of the whole cone, by the last rule; and also $p\times\frac{ph}{P-p}$=twice the convex surface of the part ECD. Therefore $P\times(h+\frac{ph}{P-p})-p\times\frac{ph}{P-p}=hP+\overline{P-p}\times\frac{ph}{P-p}=hP+hp$ $=\overline{P+p}\times h=$twice the convex surface of the frustum ABDE; and the half thereof is $\frac{(P+p)\times h}{2}$, which is the same as the rule. Q. E. D.

To find the surface of the frustum of a right pyramid.

Rule. Multiply the sum of the perimeters of the ends by the slant height, and half the product will be the surface required.

EXAMPLES.

1. In the frustum ABED, the circumferences of the two ends AB and DE are 22.5 and 15.75 respectively, and the slant height BD is 26; what is the convex surface?



And $\frac{9.4248 \times 15}{2} = \frac{141.3720}{2} = 70.686$ *square feet, the convex surface required.*

2. The diameter of a right cone is 4.5 feet, and the slant height 20 feet; required the convex surface.

Ans. 141.372 feet.

3. The circumference of the base is 10.75, and the slant height 18.25; what is the convex surface?

Ans. 98.09375.

PROBLEM VII.

To find the convex surface of the frustum of a right cone.

RULE.*

Multiply the sum of the perimeters of the two ends, by the slant height of the frustum, and half the product will be the surface required.

**Demon.* Let the perimeter of the circle AB=P, that of DE=p, BD=h, and the rest as in the last problem.

2. What is the convex surface of the frustum of a right cone, the circumference of the greater end being 30 feet, that of the less end 10 feet, and the length of the slant side 20 feet? Ans. 400 feet.

3. What is the convex surface of the frustum of a right cone, the diameters of the ends being 8 and 4 feet, and the length of the slant side 20 feet?

Ans. 376.992 feet.

4. If a segment of 6 feet slant height be cut off a cone whose slant height is 30 feet, and circumference of its base 10 feet; what is the surface of the frustum?

Ans. 144 feet.

PROBLEM VIII.

To find the solidity of a cone or pyramid.

RULE.*

Multiply the area of the base by one third of the perpendicular height of the cone or pyramid, and the product will be the solidity.

---

* *Demon.* Let SC$=a$, C$s=x$, *and* A$=$area of the base of the cone ACB.

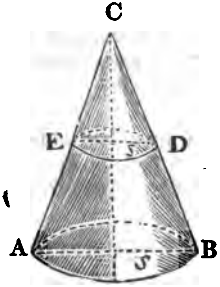
Then $a^2(\mathrm{CS}^2)$; $x^2(\mathrm{C}s^2)$: : AB^2 : ED^2 (by sim. $\triangle$s) : : A : $\frac{\mathrm{A}x^2}{a^2}$, (=area of the circle ED) because all the circles are as the squares of their diameters.

But $\frac{\mathrm{A}x^2}{a^2}\times\dot{x}=$fluxion of the cone ECD, and its fluent$=\frac{\mathrm{A}x^3}{3a^2}$, which, when $x=a$, becomes $\frac{\mathrm{A}a}{3}=\mathrm{A}\times\frac{a}{3}$ for the solidity of the whole cone. Q. E. D.

In the pyramid CEDB it will be $a^2(\mathrm{CS}^2)$: x^2 $(\mathrm{C}s^2)$: : CE^2 : ce^2 : : ED^2 : eo^2 (by sim. $\triangle$s) : : A (area of the base EB) : $\frac{\mathrm{A}x^2}{a^2}$ (area of the polygon *eb*) because all similar figures are as the squares of their like sides.

EXAMPLES.

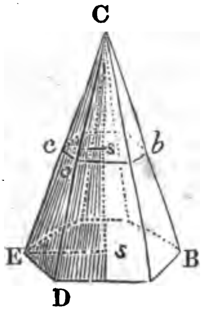
1. Required the solidity of the cone ACB, whose diameter AB is 20, and its perpendicular height CS 24.



Here $.7854\times 20^2=.7854\times 400=314.16$=*area of the base* AB.

And $314.16\times\frac{24}{3}=314.16\times 8=2513.28$ = *solidity required.*

2. Required the solidity of the hexagonal pyramid ECBD, each of the equal sides of its base being 40, and the perpendicular height CS 60.



But $\frac{Ax^2}{a^2}\times\dot{x}$=fluxion of the pyramid *ceob*, and its correct fluent$=A\times\frac{a}{3}$ the same as in the cone; and this rule is general, let the figure of the base be what it will.

Here 2.598076 (*multiplier when the side is* 1) $\times 40^2 = 2.598076 \times 1600 = 4156.9216 =$ *area of the base.*

And $4156.9216 \times \frac{60}{3} = 4156.9216 \times 20 = 83138.432$ *solidity required.*

3. Required the solidity of a triangular pyramid, whose height is 30, and each side of the base 3. Ans. 38.97117.

4. Required the solidity of a square pyramid, each side of whose base is 30, and the perpendicular height 20.
Ans. 6000.

5. What is the solidity of a cone, the diameter of whose base is 18 inches, and its altitude 15 feet?
Ans. 8.83575 feet.

6. If the circumference of the base of a cone be 40 feet, and the height 50 feet; what is the solidity?
Ans. 2122.1333 feet.

7. What is the content of a pentagonal pyramid, its height being 12 feet, and each side of its base 2 feet?
Ans. 27.5276.

PROBLEM IX.

To find the solidity of a frustum of a cone or pyramid.

RULE.*

I. *For the frustum of a cone, the diameters, or circumferences of the two ends, and the height being given.*

---

* *Demon.* First let $\mathrm{D} =$ diameter AB, $d = \mathrm{ED}$, $p = .7854$, $h = \mathrm{S}s =$ the height of the frustum ABDE of the cone. *See the last figures.*

Then $\mathrm{D} : d :: \mathrm{CS} : \mathrm{C}s$, and $\mathrm{D} - d : d :: \mathrm{CS} - \mathrm{C}s\ (h) : \frac{dh}{\mathrm{D}-d} = \mathrm{C}s =$ height of the cone EDC. But $\frac{p\mathrm{D}^2}{3} \times \left(h + \frac{dh}{\mathrm{D}-d}\right)$ = solidity of the whole cone ACB, and $\frac{pd^2}{3} \times \frac{dh}{\mathrm{D}-d}$ = the solidity of the cone ECD. Therefore $\frac{p\mathrm{D}^2}{3} \times \left(h + \frac{dh}{\mathrm{D}-d}\right)$

Add together the square of the diameter of the greater end, the square of the diameter of the less end, and the product of the two diameters; multiply the sum by .7854, and the product by the height; $\frac{1}{3}$ of the last product will be the solidity. Or,

Add together the square of the circumference of the greater end, the square of the circumference of the less end, and the product of the two circumferences; multiply the sum by .07958 and the product by the height; $\frac{1}{3}$ of the last product will be the solidity.

II. *For the frustum of a pyramid whose sides are regular polygons.*

Add together the square of a side of the greater end, the square of a side of the less end, and the product of

---

$$-\left(\frac{pd^2}{3}\times\frac{dh}{\mathrm{D}-d}\right)=\left(\mathrm{D}^2\times\frac{\mathrm{D}h}{\mathrm{D}-d}-d^2\times\frac{dh}{\mathrm{D}-d}\right)\frac{p}{3}=\left(hd^2+\overline{\mathrm{D}^2-d^2}\times\frac{\mathrm{D}h}{\mathrm{D}-d}\right)\frac{p}{3}=\frac{\mathrm{D}^3-d^3}{\mathrm{D}-d}\times\frac{hp}{3}=(\mathrm{D}^2+d^2+\mathrm{D}d)\times\frac{hp}{3}=$$

the solidity of the frustum ABDE, which is the same as the rule.

And, since the circumferences of circles have the same ratio that their diameters have, if C be put for the circumference of the greater end, c=that of the less end, and p=.07958, the demonstration of the rule, when the circumferences are given, will differ in nothing from the above.

Again, for the polygon, let S=ED, s=ed, and m=proper multiplier in the table of polygons; then S : s : : CS : Cs, and S—s : s : : CS—Cs (h) : $\frac{hs}{\mathrm{S}-s}$

But $m\mathrm{S}^2$ and ms^2 are the areas of polygons whose sides are S and s respectively. And therefore $\frac{m\mathrm{S}^2}{3}+\left(h+\frac{hs}{\mathrm{S}-s}\right)$

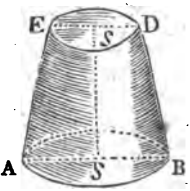
$$-\frac{ms^2}{3}\times\frac{hs}{\mathrm{S}-s}=(m\mathrm{S}^2+\overline{m\mathrm{S}^2-ms^2}\times\frac{s}{\mathrm{S}-s}\times\frac{h}{3}=(m\mathrm{S}^2+ms^2+m\mathrm{S}s)\times\frac{h}{3}=(\mathrm{S}^2+s^2+\mathrm{S}s)\times\frac{mh}{3}=$$ solidity of the frustum $e$EDB$b$ which is the same as the rule.

these two sides; multiply the sum by the proper number in the table, Prob. VIII. of Superficies, and the product by the height: $\frac{1}{3}$ of the last product will be the solidity.

Note.—When the ends of the pyramids are not regular polygons. Add together the areas of the two ends and the square root of their product; multiply the sum by the height, and $\frac{1}{3}$ of the product will be the solidity.

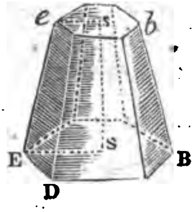
EXAMPLES.

1. What is the solidity of the frustum of the cone EABD, the diameter of whose greater end AB is 5 feet, that of the less end ED, 3 feet, and the perpendicular height Ss, 9 feet?



$$\frac{(5^2+3^2+\sqrt{5\times3})\times.7854\times9}{3}=\frac{346.3614}{3}=115.4538$$ *solid feet, the content of the frustum.*

2. What is the solidity of the frustum *e*EDB*b* of an hexagonal pyramid, the side ED of whose greater end is 4 feet, that *eb* of the less end 3 feet, and the height S*s*, 9 feet?



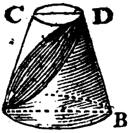
$$\frac{(4^2+3^2+\overline{4\times3})\times 2.598076\times 9}{3}=\frac{865.159308}{3}=288.386436$$

solid feet, the solidity required.

3. What is the solidity of the frustum of a cone, the diameter of the greater end being 4 feet, that of the less end 2 feet, and the altitude 9 feet? Ans. 65.9736.

The following cases contain all the rules for finding the superficies and solidities of conical ungulas.

I. *When the section passes through the opposite extremities of the ends of the frustum.*



Let $D=AB$ the diameter of the greater end; $d=CD$, the diameter of the less end; $h=$perpendicular height of the frustum, and $n=.7854$.

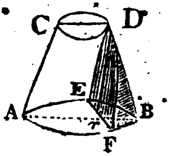
Then $\frac{D^2-d\sqrt{Dd}}{D-d}\times\frac{nDh}{3}=$ solidity of the greater elliptic ungula ADB.

$\frac{D\sqrt{Dd}-d^2}{D-d}\times\frac{ndh}{3}=$ solidity of the less ungula ACD.

$\frac{(D^{\frac{3}{2}}-d^{\frac{3}{2}})^2}{D-d}\times\frac{nh}{3}=$ difference of these hoofs.

And $\frac{n}{D-d}\sqrt{4h^2+(D-d^2)}\times(D^2-\frac{D+d}{2}\sqrt{Dd})=$ curve surface of ADB.

II. *When the section cuts off part of the base, and makes the angle* DrB *less than the angle* CAB.



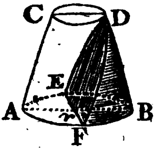
4. What is the solidity of the frustum of a cone, the circumference of the greater end being 40, that of the less end 20, and the length or height 50? Ans. 3713.7333.

Let S = tabular segment, whose versed sine is $\mathrm{B}r \div \mathrm{D}$, s — tab. seg. whose versed sine is $\overline{\mathrm{B}r - (\mathrm{D} - d)} \div d$, and the other letters as above.

Then $\left(\mathrm{S} \times \mathrm{D}^3 - s \times d^3 \times \frac{\mathrm{B}r}{\mathrm{B}r - \overline{\mathrm{D} - d}} \sqrt{\frac{\mathrm{B}r}{\mathrm{B}r - \overline{\mathrm{D} - d}}}\right) \times \frac{\frac{1}{2}h}{\mathrm{D} - d} =$ solidity of the elliptic hoof EFBD.

And $\frac{1}{\mathrm{D} - d} \sqrt{4h^2 + (\mathrm{D} - d)^2} \times \left(\text{seg. FBE} - \frac{d^2}{\mathrm{D}^2} \times \frac{\frac{1}{2} \times (\mathrm{D} + d) - \mathrm{A}r}{d - \mathrm{A}r} \times \sqrt{\frac{\mathrm{B}r}{d - \mathrm{A}r}} \times \right.$ seg. of the circle AB, whose height is $\left. \mathrm{D} \times \frac{d - \mathrm{A}r}{d} \right) =$ equal convex surface of EFBD.

III. *When the section is parallel to one of the sides of the frustum.*



Let A = area of the base FBE, and the other letters as before.

Then $\left(\frac{\mathrm{A} \times \mathrm{D}}{\mathrm{D} - d} - \frac{4}{3} d \sqrt{(\mathrm{B} - d) \times d}\right) \times \frac{1}{3} h =$ solidity of the parabolic hoof EFBD.

And $\frac{1}{\mathrm{D} - d} \sqrt{4h^2 + (\mathrm{D} - d)^2} \times \left(\text{seg. FBE} - \frac{2}{3} \overline{\mathrm{D} - d} \times \sqrt{d \times \overline{\mathrm{D} - d}}\right) =$ convex surface of EFBD.

IV. *When the section cuts off part of the base, and makes the angle* D*r*B *greater than the angle* CAB.

5. What is the solidity of the frustum of a square pyramid, one side of the greater end being 18 inches, that of the less end 15 inches, and the height 60 inches?

Ans. 16380 inches.

6. What is the solidity of the frustum of an hexagonal pyramid, the side of whose greater end is 3 feet, that of the less end 2 feet, and the length 12 feet?

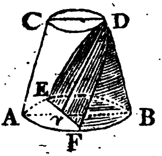
Ans. 197.453776 feet.

PROBLEM X.

To find the solidity of a cuneus or wedge.

RULE.*

Add twice the length of the base to the length of the edge, and reserve the number.



Let the area of the hyperbolic section EDF=A, and the area of the circular seg. EBF=a.

Then $\frac{\frac{1}{3}h}{\text{D}-h} \times \left(a \times \text{D} - \text{A} \times \frac{d \times \text{E}r}{\text{C}r}\right)$ = solidity of the hyperbolic ungula EFBD.

And $\frac{1}{\text{D}-d} \times \sqrt{4h^2+(\text{D}-d)^2} \times \left(\text{cir. seg. EBF} - \frac{d^2}{\text{D}^2} \times \frac{\text{B}r - \frac{1}{2}(\text{D}-d)}{\text{B}r-\text{D}-d} \sqrt{\frac{\text{B}r}{\text{B}r-d-\text{D}}}\right)$ = curve surface of EFBD.

Note.—The transverse diameter of the hpy. seg. = $\frac{d \times \text{C}r}{\text{D}-d-\text{B}r}$ and the conjugate = $d\sqrt{\frac{\text{B}r}{\text{D}-d-\text{B}r}}$, from which its area may be found by the former rules.

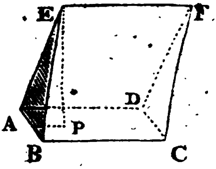
---

* *Demon.* When the length of the base is equal to half of the wedge, the wedge is evidently equal to half a prism of the same base and altitude.

Multiply the height of the wedge by the breadth of the base, and this product by the reserved number; $\frac{1}{6}$ of the last product will be the solidity.[*]

EXAMPLES.

1. How many solid feet are there in a wedge, whose base is 5 feet 4 inches long, and 9 inches broad, the length of the edge being 3 feet 6 inches, and the perpendicular height 2 feet 4 inches?



Here $\frac{(64\times2+42)\times28\times9}{6} = \frac{(128+42)\times28\times9}{6} = \frac{170\times28\times9}{6} = \frac{170\times28\times3}{2} = 170\times14\times3 = 7140$ *solid inches.*

And $7140\div1728 = 4.1319$ *solid feet, the content required.*

---

[*] And according as the edge is shorter or longer than the base, the wedge is greater or less than half a prism, by a pyramid of the same height and breadth at the base with the wedge, and the length of whose base is equal to the difference of the lengths of the edge and base of the wedge.

Therefore, let the length of the base BC=L; the length of the edge EF=l; the breadth of the base BA=b; and the height of the wedge EP=h; and we shall have by the former rules

$\frac{blh}{2}\pm bh\times\frac{\pm L\pm l}{3}=\frac{blh}{2}+bh\times\frac{L-l}{3}=bh\times\frac{3l+2L-2l}{6}=bh\times\frac{2L+l}{6}$. Q. E. D.

2. The length and breadth of the base of a wedge are 35 and 15 inches, and the length of the edge is 55 inches: what is the solidity, supposing the perpendicular height to be 17.14508 inches? Ans. 3.1006 feet.

PROBLEM XI.

To find the solidity of a prismoid.

RULE.*

To the sum of the areas of the two ends add four times the area of a section parallel to and equally distant from both ends, and this last sum multiplied by $\frac{1}{6}$ of the height will give the solidity.

Note. The length of the middle rectangle is equal to half the sum of the lengths of the rectangles of the two ends, and its breadth equal to half the sum of the breadths of those rectangles.

---

* *Demon.* The rectangular prismoid is evidently composed of two wedges, whose heights are equal to the height of the prismoid, and their bases its two ends. Wherefore, by the last problem its solidity will be $=(\overline{2L+l}\times B+\overline{2l+L}\times b)\times\frac{h}{6}$, which, by putting $M=\frac{L+l}{2}$ and $m=\frac{B+b}{2}$ becomes $BL+bl+4Mm\times\frac{h}{6}$; which is the rule, as was to be shown.

The solidities of the two parts, commonly called the ungulas, or hoofs, into which the frustum of a rectangular pyramid is divided, may be found by the last two rules, as they are only composed of wedges and prismoids.

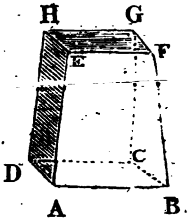
A very elegant demonstration of this rule for the prismoid may be seen in *Simson's Fluxions, p.* 179, *2d Edition.*

If the bases of the prismoid are dissimilar rectangles, of which L, l and M, m are corresponding dimensions, and h the height;

Then $(\overline{2L+l}.M+\overline{2l+L}.m)\times\frac{1}{6}h=$solidity.

EXAMPLES.

1. What is the solidity of a rectangular prismoid, the length and breadth of one end being 14 and 12 inches, and the corresponding sides of the other 6 and 4 inches, and the perpendicular 30½ feet.



Here $14\times12+\overline{6\times4}=168+24=192=$ *sum of the areas of the two ends.*

Also $\frac{14+6}{2}=\frac{20}{2}=10=$ *length of the middle rectangle.*

And $\frac{12+4}{2}=\frac{16}{2}=8=$ *breadth of the middle rectangle.*

Whence $10\times8\times4=80\times4=320=4$ *times the area of the middle rectangle.*

Or $(320+192)\times\frac{366}{6}=512\times61=31232$ *solid inches.*

And $31232\div1728=18.074$ *solid feet, the content.*

2. What is the solid content of a prismoid, whose greater end measures 12 inches by 8, the less end 8 inches by 6, and the length, or height, 60 inches? Ans. 2.453 feet.

3. What is the capacity of a coal wagon, whose inside dimensions are as follow: at the top, the length is 81½, and breadth 55 inches; at the bottom the length is 41, and the breadth 29½ inches; and the perpendicular depth is 47½ inches? Ans. 126340.59375 cubic inches.

PROBLEM XII.

To find the convex surface of a sphere.

RULE.*

Multiply the diameter of the sphere by its circumference, and the product will be the convex superficies required.

Note.—The curve surface of any zone or segment will also be found by multiplying its height by the whole circumference of the sphere.

---

* *Demon.* Put the diameter BG$=d$, BA$=x$, AC$=y$, BC$=z$; and 3.1416$=p$.

Then, since the triangles AOC and CED are similar, we shall have CA (y) : CO $\frac{d}{2}$ :: CE $(\dot{x})$: CD $(\dot{z})=\frac{d\dot{x}}{2y}$. But $2py\dot{z}$ is the general expression for the fluxion of any surface; and therefore, by substituting $\frac{d\dot{x}}{2y}$ for its equal $\dot{z}$, the fluxion will become $pd\dot{x}$; and consequently pdx=surface of any segment of a sphere whose height is x, and pdd=that of the whole sphere. Q. E. D.

Cor. 1. The surface of a sphere is also equal to the curve surface of its circumscribing cylinder.

Cor. 2. The surface of a sphere is also equal to four times the area of a great circle of it.

1. To find the lunar surface included between two great circles of the sphere.

Rule. Multiply the diameter into the breadth of the surface in the middle, and the product will be the superficies required. Or,

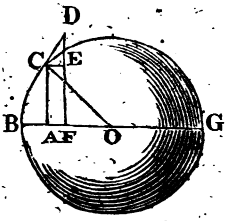
As one right angle is to a great circle of the sphere;
So is the angle made by the two great circles,
To the surface included by them.

2. To find the area of a spherical triangle, or the surface included by the intersecting arcs of three great circles of the sphere.

Rule. As two right angles, or 180°,
Is to a great circle of the sphere;
So is the excess of the three angles above two right angles,
To the area of the triangle.

EXAMPLES.

1. What is the convex superficies of a globe BCG, whose diameter BG is 17 inches?



Here 3.1416 × 17 × 17=53.4072 × 17=907.9224 *square inches.*

And 907.9224 ÷ 144 = 6.305 *square feet, the answer.*

2. What is the convex superficies of a sphere whose diameter is $1\frac{1}{3}$ feet, and the circumference 4.1888 feet? Ans. 5.58506 feet.

3. If the diameter, or axis of the earth be $7957\frac{3}{4}$ miles, what is the whole surface, supposing it to be a perfect sphere? Ans. 198944286.35235 sq. miles.

4. The diameter of a sphere is 21 inches; what is the convex superficies of that segment of it whose height is $4\frac{1}{2}$ inches? Ans. 296.8812 inches.

5. What is the convex surface of a spherical zone, whose breadth is 4 inches, and the diameter of the sphere, from which it was cut, 25 inches? Ans. 314.16 inches.

PROBLEM XIII.

To find the solidity of a sphere or globe.

RULE.*

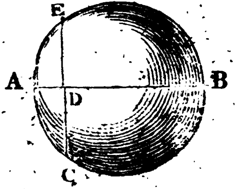
Multiply the cube of the diameter by .5236, and the product will be the solidity.

---

* *Demon.* Put AD $=x$, DC $=y$, the diameter AB $=d$, and $p=3.146$.

EXAMPLES.

1. What is the solidity of the sphere AEBC, whose diameter AB is 17 inches?



Here $17^3\times.5236=17\times17\times17\times.5236=289\times17\times.5236=4913\times.5236=2572.4468$ *solid inches.*

And $2572.4468\div1728=1.48868$ *solid feet, the answer.*

2. What is the solidity of a sphere whose diameter is $1\frac{1}{3}$ feet? Ans. 1.2411 feet.

3. What is the solidity of the earth, supposing it to be perfectly spherical, and its diameter $7957\frac{3}{4}$ miles?
Ans. 263858149120 miles.

Then, by the property of the circle, $dx-x^2=y^2$. But the general expression for the fluxion of any solid is $py^2\dot{x}$; and therefore by writing $dx-x^2$ for its equal y^2, we shall have $p\dot{x}\times\overline{dx-x^2}=pdx\dot{x}-px^2\dot{x}$. The fluent of which is $\frac{pdx^2}{2}-\frac{px^3}{3}=\frac{3pdx^2-2px^3}{6}=$ content of the segment CAE. And if d be substituted for x, it will become $\frac{3pd^3-2pd^3}{6}=\frac{pd^3}{6}=d^3\times.5236$, or $.5236d^3$; which is the same as the rule.

Coroll. A sphere, or globe, is equal to two-thirds of its circumscribing cylinder.

PROBLEM XIV.

To find the solidity of the segment of a sphere.

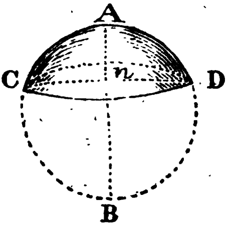
RULE.*

To three times the square of the radius of its base add the square of its height; and this sum multiplied by the height, and the product again by .5236, will give the solidity. Or,

From three times the diameter of the sphere subtract twice the height of the segment, multiply by the square of the height, and that product by .5236; the last product will be the solidity.

EXAMPLES.

1. The radius C*n* of the base of the segment CAD is 7 inches, and the height A*n* 4 inches; what is its solidity?



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* *Demon.* Let r = radius of the base of the segment, h = height of the segment, and the other letters as before.

Then will $(3dh^2-2h^3)\times\frac{p}{6}$ = solidity of the segment, as is shown in the last problem.

But since $\frac{r^2+h^2}{h}=d$, by the property of the circle, we shall

Here $(7^2\times3+4^2)\times4\times.5236=(49\times3+4^2)\times4\times.5236=(147+4^2)\times4\times.5236=(147+16)\times4\times.5236=163\times4\times.5236=652\times.5236=341.3872$ solid inches, the answer.

2. What is the solidity of the segment of a sphere, the diameter of whose base is 20, and its height 9?

Ans. 1795.4244.

3. What is the content of the spherical segment, whose height is 4 inches, and the radius of its base 8?

Ans. 435.6352.

4. What is the solidity of a spherical segment, the diameter of its base being 17.23368, and its height 4.5?

Ans. 572.5566.

5. The diameter of a sphere being 6 inches, required the solidity of the segment whose altitude is 2 inches.

Ans. 29.3216 cubic inches.

6. Required the solidity of a spherical segment, the height of which is 15, the diameter of the sphere being 18.

Ans. 2827.44.

PROBLEM XV.

To find the solidity of a frustum or zone of a sphere.

RULE.*

To the sum of the squares of the radii of the two ends, add one third of the square of their distance, or of the

---

have $(\frac{3r^2h^2\times3h^4}{h}-2h^3)\times\frac{p}{6}=\overline{3r^2+h^2}\times\frac{ph}{6}=$ solidity of the segment, which is the same as the rule.

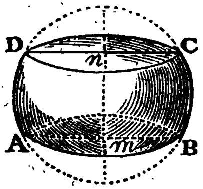
Or, if $d=$ diameter of the sphere, and $h=$ height of the segment; then will $.5236h^2\times(3d-2h)=$ solidity.

* *Demon.* The difference between two segments of a sphere whose heights are H and h, and the radii of whose bases are R and r, will, by the last problem $=\frac{p}{6}\times(3R^2H+H^3-3r^2h-h^3)$ $=$ zone whose height is $H-h$. And therefore by putting a for

breadth of the zone, and this sum multiplied by the said breadth, and the product again by 1.5708, will give the solidity.

EXAMPLES.

1. What is the solid content of the zone ABCD, whose greater diameter AB is 20 inches, the less diameter CD 15 inches, and the distance *nm* of the two ends 10 inches?



Here $(10^2+7.5^2+\frac{10^2}{3})\times10\times1.5708=(100+56.25+33.33)\times10\times1.5708=189.58\times10\times1.5708=1895.8\times1.5708=2977.92264$ *solid inches, the answer.*

2. What is the solid content of a zone, whose greater diameter is 24 inches, the less diameter 20 inches, and the distance of the ends 4 inches? Ans. 1566.6112 inches.

3. Required the solidity of the middle zone of a sphere, whose top and bottom diameters are each 3 feet, and the breadth of the zone 4 feet? Ans. 61.7848 feet.

---

the altitude of the frustum, and exterminating H and h by the means of the two equations $\frac{R^2+H^2}{H}=\frac{r^2+h^2}{h}$ and $H-h=a$, we shall have $(R^2+r^2+\frac{a^2}{3})\times\frac{pa}{2}$, which is the rule.

If it be the middle zone of the sphere, the solidity will be= $d^2+\frac{2}{3}h^2)\times .7854h$; where d=diameter of each end, and h= its height,

PROBLEM XVI.

To find the solidity of a spheroid.

RULE.*

Multiply the square of the revolving axe by the fixed axe, and this product again by .5236, and it will give the solidity required.

Where note that .5236 is $=\frac{1}{6}$ of 3.1416.

EXAMPLES.

1. In the prolate spheroid ABCD, the transverse, or fixed axe AC is 90, and the conjugate, or revolving axe DB is 70: what is the solidity?

---

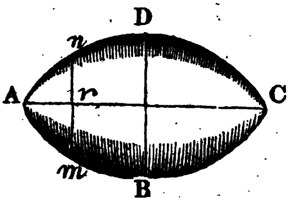
* *Demon.* Let AC $=a$, DB $=b$, A$r=x$, $rn=y$, and $p=$ 3.14159, &c.

Then $a^2 : b^2 :: x\times(a-x) : \frac{b^2}{a^2}\times(ax-x^2)=y^2$ by the property of the ellipsis.

And therefore the fluxion of the solid $(=py^2\dot{x})=\frac{pb^2}{a^2}\times(ax\dot{x}-x^2\dot{x})$; and its fluent $\times\frac{pb^2}{a^2}\times(\frac{1}{2}ax^2-\frac{1}{3}x3)=$ segment nAm. Which, when $x=a$ becomes $\frac{pb^2}{a^2}\times(\frac{1}{2}a^3-\frac{1}{3}a^3)=\frac{pab^2}{6}=$ content of the whole spheroid. Q. E. D.

If f be put $=$ fixed axe, $r=$ revolving axe, $q=(f^2\backsim r^2)\div f^2$, and $p=3.1416$, &c.

Then will $prf\sqrt{1+\frac{1}{3}q}=$ surface of the oblate spheroid, and $prf\sqrt{1-\frac{1}{3}q}=$ that of the prolate spheroid.



Here $DB^2 \times AC \times .5236 = 70^2 \times 90 \times .5236 = 4900 \times 90 \times .5236 = 441000 \times .5236 = 230907.6 =$ *solidity required.*

2. What is the solidity of a prolate spheroid, whose fixed axe is 100, and its revolving axe 60? Ans. 188496.

3. What is the solidity of an oblate spheroid, whose fixed axe is 60, and its revolving axe is 100? Ans. 314160.

PROBLEM XVII.

To find the content of the middle frustum of a spheroid, its length, the middle diameter, and that of either of the ends being given.

CASE I.

When the ends are circular, or parallel to the revolving axis.

RULE.*

To twice the square of the middle diameter add the square of the diameter of either of the ends, and this sum multiplied by the length of the frustum, and the product again by .2618, will give the solidity.

---

* *Demon.* Let AO $= a$, DO $= b$, EN $= h$, NO $= c$, RO $= x$, RE $= y$, and $p = 3.14159$, &c.

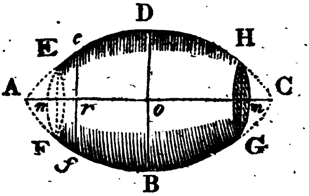
Then $a^2 : b^2 :: a^2 - x^2 : \frac{b^2}{a^2} \times (a^2 - x^2) = b^2 - \frac{b^2 x^2}{a^2} = y^2$ by the property of the ellipsis.

And also $a^2 : b^2 :: \overline{a^2 - c^2} : \frac{b^2 \times (a^2 - c^2)}{a^2} = h^2$; or $a^2 = b^2 c^2 \div (b^2 - h^2.)$

Where note that $.2618 = \frac{1}{12}$ of 3.1416.

EXAMPLES.

1. In the middle frustum of a spheroid EFGH, the middle diameter DB is 50 inches, and that of either of the ends EF or GH is 40 inches, and its length *nm* 18 inches: what is its solidity?



Here $(50^2 \times 2 + 40^2) \times 18 \times .2618 = (2500 \times 2 + 1600) \times 18 \times .2618 = (5000 + 1600) \times 18 \times .2618 = 6600 \times 18 \times .2618 = 118800 \times .2613 = 31101.84$ cubic inches, the answer.

2. What is the solidity of the middle frustum of a prolate spheroid, the middle diameter being 60, that of either of the two ends 36, and the distance of the ends 80?

Ans. 177940.224.

Whence, by substituting this value of a^2 in the former equation, we shall have $y^2 = b^2 - \frac{b^4x^2 - b\ h^2x^3}{b^2c^2} = b^2 - \frac{b^2x^2 - h^2x^2}{c^2} = b^2 - \frac{x^2}{c^2} \times (b^2 - h^2.)$

And consequently the fluxion of the solid $(py^2\dot{x}) = pb^2\dot{x} - \frac{px^2\dot{x}}{c^2} \times (b^2 - h^2)$; the fluent of which is $= pb^2x - \frac{px^3}{3c^2} \times (b^2 - h^2)$; which, when $x = c$, becomes $pb^2c - \frac{pcb^2 - pch^2}{3} = \frac{pc \times (2b^2 + h^2)}{3} = \frac{pc}{12} \times \overline{8b^2 + 4b^2}$. Q. E. D.

3. What is the solidity of the middle frustum of an oblate spheroid, the middle diameter being 100, that of either of the ends 80, and the distance of the ends 36?

Ans. 248814.72.

CASE II.

When the ends are elliptical or perpendicular to the revolving axis.

RULE.*

1. Multiply twice the transverse diameter of the middle section by its conjugate diameter, and to this product add the product of the transverse and conjugate diameters of either of the ends.

---

* *Demon.* Put $so=a$, $EO=b$, $om=r$, $on=x$, $An=y$, $xc=z$, and $p=3.14159$, &c.

Then $a^2 : b^2 :: a^2-x^2 : \frac{b^2}{a^2}\times(a^2-x^2)=y^2$ by the property of the ellipsis.

And, since ACD is an ellipsis similar to EmF, it will be $b : r :: y : \frac{ry}{b}=z$; as is shown by the writers on Conics.

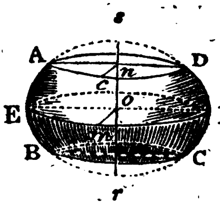
But the fluxion of the solid AEFD is $py\dot{z}x=py\dot{x}\times\frac{ry}{b}=\frac{pry^2\dot{x}}{b}=\frac{pr\dot{x}}{b}\times\frac{b^2\times(a^2-x^2)}{a^2}=prb\dot{x}\times\frac{a^2-x^2}{a^2}$. And the fluent $=prbx\times\frac{a^2-\frac{1}{3}x^2}{a^2}$. Which, by substituting for a^2 its value $\frac{b^2x^2}{b^2-y^2}$ becomes $=prx\times\frac{2b^2+y^2}{3b}=px\times\frac{2}{3}rb+\frac{ry^2}{3b}$.

And this again, by putting z for its equal $\frac{ry}{b}$, becomes $=\frac{px}{3}\times\overline{2rb+yz}=$ frustum EFDA. Or $\frac{p\times ne}{12}\times(2EF\times2om+AD\times2ne)=$ middle frustum ABCD. Q. E. D.

2. Multiply the sum thus found, by the distance of the ends or the height of the frustum, and the product again by .2618, and it will give the solidity required.

EXAMPLES.

1. In the middle frustum ABCD of an oblate spheroid, the diameters of the middle section EF are 50 and 30; those of the end AD 40 and 24; and its height *ne* 18; what is the solidity?



Here $(50\times2\times30+\overline{40\times24})\times18\times.2618=(3000+960)\times18\times.2618=3960\times18\times.2618=71280\times.2618=$ $18661.104=$ *solidity required.*

2. In the middle frustum of a prolate spheroid, the diameters of the middle section are 100 and 60; those of the end 80 and 48; and the length 36: what is the solidity?
Ans. 149288.832.

3. In the middle frustum of an oblate spheroid, the diameters of the middle section are 100 and 60; those of the end 60 and 36; and the length 80: what is the solidity of the frustum?
Ans. 296567.04.

PROBLEM XVIII.

To find the solidity of the segment of a spheroid.

CASE I.

When the base is parallel to the revolving axis.

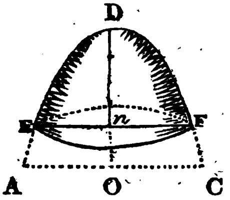
RULE.*

1. Divide the square of the revolving axis by the square of the fixed axe, and multiply the quotient by the difference between three times the fixed axe and twice the height of the segment.

2. Multiply the product, thus found, by the square of the height of the segment, and this product again by .5236, and it will give the solidity required.

EXAMPLES.

1. In the prolate spheroid DEFD, the transverse axis 2 DO is 100, the conjugate AC 60, and the height Dn of the segment EDF 10; what is the solidity?



Here $\left(\frac{60^2}{100^2} \times \overline{300-20}\right) \times 10^2 \times .5236 = .36 \times 280 \times 10^2 \times .5236 = 100.80 \times 100 \times .5236 = 10080 \times .5236 = 5277.888$ = *solidity required.*

2. The axes of a prolate spheroid are 50 and 30; what is the solidity of that segment whose height is 5, and its base perpendicular to the fixed axe?

Ans. 659.736.

---

* This rule is formed from the theorem for the segment in the demonstration to problem XX.

The content of the segment may also be found by the following theorem:

$(D^2+4d^2) \times \frac{1}{6}nh$ = content of the segment; D being the diameter of the base, d = diameter in the middle, h = height, and n = .7854 = area of a circle whose diameter is 1.

3. The diameters of an oblate spheroid are 100 and 60; what is the solidity of that segment whose height is 12, and its base perpendicular to the conjugate axe?

Ans. 32672.64.

CASE II.

When the base is perpendicular to the revolving axis.

RULE.*

1. Divide the fixed axe by the revolving axe, and multiply the quotient by the difference between three times the revolving axe and twice the height of the segment.

2. Multiply the product, thus found, by the square of the height of the segment, and this product again by .5236, and it will give the solidity required.

EXAMPLES.

1. In the prolate spheroid $a\mathrm{E}b\mathrm{F}$, the transverse axe EF is 100, the conjugate ab 60, and the height an of the segment $a\mathrm{AD}$ 12; what is the solidity?

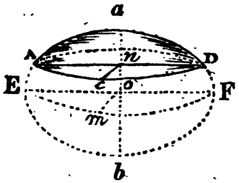
---

* *Demon.* Put $ao=a$, $\mathrm{E}o=b$, $om=r$, $an=x$, $\mathrm{A}n=y$, $nc=z$, and $p=3.1416$. Then will $a^2 : b^2 :: a^2-(a-x)^2$ or $(2ax-x^2) : \frac{b^2\times(2ax-x^2)}{a^2}=y^2$ by the property of the ellipse.

And since ACD is an ellipse similar to EMF, it will be $b : r :: y : \frac{ry}{b}=z$; as is shown by the writers on Conics.

But the fluxion of the solid $a\mathrm{ACD} = pyz\dot{x} = py\dot{x} \times \frac{ry}{b}=\frac{pry^2\dot{x}}{b}=\frac{pr\dot{x}}{b}\times\frac{b^2\times(2ax-x^2)}{a^2}$ whose fluent is $=\frac{prb}{a}x^2-\frac{prb}{3a^2}x^3$; which, when $x=h=$the height of the segment, becomes $(3ah^2-h^3)\times\frac{prb}{3a^2}$. Whence, since $r=a$, we shall have $(3ab^2-h^3)\times\frac{pb}{3a}=$solidity of the segment.

Q. E. D.



Here 156 (=*diff. of* 3*ab and* 2*an*) ×1⅔ (=EF÷*ab*× 144 (=*square of an*)×.5236=$\frac{156\times5}{3}$×144×.5236=52 ×5×144×.5236=260×144×.5236=37440 × .5236= 19603.584=*solidity required.*

2. Required the content of the segment of a prolate spheroid: its height being 6, and the axes 50 and 30.

Ans. 2450.448.

PROBLEM XIX.

To find the solidity of a parabolic conoid.

RULE.*

Multiply the area of the base by half the altitude, and the product will be the content.

---

* *Demon.* Let DM$=a$, BM$=b$, DN$=x$, EN$=y$, and $p=$ 3.1416.

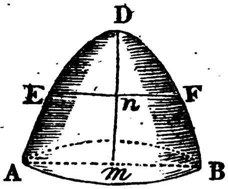
Then, by the nature of the parabola $a : b^2 :: x : y^2$, *or* $\frac{b^2x}{a}=y^2$; wherefore $\frac{pb^2x\dot{x}}{a}$ $(=py^2\dot{x})$ = the fluxion of the solid, and $\frac{pb^2x^2}{2a}$ = its fluent; which, when x becomes $=a$, is $\frac{1}{2}pab^2$ for the whole solid, or for any segment whose height is $=a$, and the radius of its base $=b$. Q. E. D.

Coroll. The parabolic conoid is =½ its circumscribing cylinder.

Note.—The rule given above will hold for any segment of the paraboloid, whether the base be perpendicular or oblique to the axe of the solid.

EXAMPLES.

1. What is the solidity of the paraboloid ADB, whose height D*m* is 84, and the diameter BA of its circular base 48?



Here $48^2\times.7854\times42(=\frac{1}{2}\,Dm)=2304\times.7854\times42=1809.5616\times42=76001.5872=$ *solidity required.*

2. What is the solidity of a paraboloid, whose height is 60, and the diameter of its circular base 100? Ans. 235620.

3. Required the solidity of a paraboloid conoid, whose height is 30, and the diameter of its base 40? Ans. 18849.6.

4. Required the solidity of a paraboloid conoid, whose height is 60, and the diameter of its base 100?
Ans. 196350.

PROBLEM XX.

To find the solidity of the frustum of a paraboloid, when its ends are perpendicular to the axe of the solid.

RULE.*

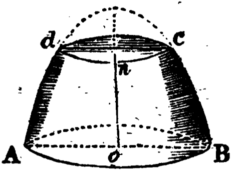
Multiply the sum of the squares of the diameters of the two ends by the height of the frustum, and the product again by .3927, and it will give the solidity.

---

* *Demon.* The segment whose base is B, and altitude A, is $=\frac{1}{2}AB$, and that whose base is b and altitude a, is $=\frac{1}{2}ab$, by the last problem: wherefore the frustum, or the difference of the seg-

EXAMPLES.

1. Required the solidity of the parabolic frustum ABCd, the diameter AB of the greater end being 58, that of the less end dc 30, and the height no 18.



Here $(58^2+30^2)\times18\times.3927=(3364+900)\times18\times.3927=4264\times18\times.3927=76752\times.3927=30140.5104=$ *solidity required.*

2. What is the solidity of the frustum of a parabolic conoid, the diameter of the greater end being 60, that of the less end 48, and the distance of the ends 18?

Ans. 41733.0144.

PROBLEM XXI.

To find the solidity of an hyperboloid.

RULE.*

To the square of the radius of the base add the square of the middle diameter between the base and the vertex; and this sum multiplied by the altitude, and the product again by .5236, will give the solidity.

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ment is $\frac{1}{2}\,\mathrm{AB}-\frac{1}{2}ab$. But $\mathrm{B}-b:\mathrm{A}-a\ (d)::\mathrm{B}:\mathrm{A}=\frac{\mathrm{B}d}{\mathrm{B}-d}$; and $\mathrm{B}-b:d::b:a=\frac{bd}{\mathrm{B}-b}$ by the nature of the paraboloid; and these values of A and a being substituted for them will make $\frac{1}{2}\,\mathrm{AB}-\frac{1}{2}\,ab=\frac{d\mathrm{B}^2-db^2}{2\mathrm{B}-2b}=\frac{1}{2}d\times(\mathrm{B}+b)$ which is the same as the rule. Q. E. D.

* *Demon.* Let $t=$ transverse, and $c=$ conjugate diameter of the generating hyperbola, $p=3.1416$, y, Y, the or-

EXAMPLES.

1. In the hyperboloid ACB, the altitude *Cr* is 10, the radius *Ar* of the base 12, and the middle diameter *nm* 15.8745; what is the solidity?

---

dinates, or semi-diameters of the ends of any frustum of the hyperboloid, $x=$ its altitude, and $\mathrm{A}=$distance of the less ordinate y from the vertex of the whole solid.

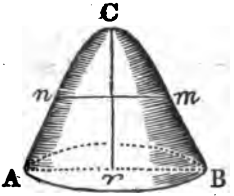
Then since $\mathrm{Y}^2=\frac{(t+\mathrm{A}+x)\times(\mathrm{A}+x)}{t^2}\times c^2$, we shall have the fluxion of the solid $=p\mathrm{Y}^2\dot{x}=pc^2\dot{x}\times\frac{\mathrm{A}t+\mathrm{A}^2+2\mathrm{A}x+tx+x^2}{t^2}$, and its fluent $=pc^2x\times\frac{\mathrm{A}t+\mathrm{A}^2+\mathrm{A}x+\frac{1}{2}tx+\frac{1}{3}x^2}{t^2}$; and this, by substituting $\frac{y^2}{c^2}$ for $\frac{\mathrm{A}t+\mathrm{A}^2}{t^2}$, and $\frac{\mathrm{Y}^2}{c^2}$ for $\frac{\mathrm{A}t+\mathrm{A}^2+2\mathrm{A}x+tx+x^2}{t^2}$ becomes $(\mathrm{Y}^2+y^2-\frac{c^2x^2}{3t^2})\times\frac{1}{2}px=$ solidity of the frustum.

But to convert this into the rules given in the text, let D, δ, d, be the greatest, middle, and least diameters, $x=$ abscissa whose ordinate is δ, and $a=$altitude. Then we shall have these three equations:

$$t^2\delta^2=c^2\times\overline{t+x}\times x$$
$$t^2d^2=c^2\times\overline{t+x-\tfrac{1}{2}a}\times\overline{x-\tfrac{1}{2}a}$$
$$t^2\mathrm{D}^2=c^2\times\overline{t+x+\tfrac{1}{2}a}\times\overline{x+\tfrac{1}{2}a}$$

From the sum of the two latter of which subtract the double of the former, and there will result $t^2\times\overline{\mathrm{D}^2-2\delta^2+d^2}=\frac{1}{2}a^2c^2$: and hence $\frac{a^2c^2}{3t^2}=\frac{2\mathrm{D}^2-4\delta^2+2d^2}{3}$. Which being substituted for it in the theorem above will give $\frac{\mathrm{D}^2+4\delta+d^2}{6}\times ap$ for the content of the frustum; which is the same as the following rule given in the text.

And if d the least diameter be supposed to become in-



Here $\overline{15.8745^2+12^2}\times 10\times .5236=\overline{251.99975+144}\times 10\times .5236=395.99975\times 10\times .5236=3959.9975\times 5236=2073.454691=$ *solidity required.*

2. In an hyperboloid the altitude is 50, the radius of the base 52, and the middle diameter 68; what is the solidity?

Ans. 191847.

PROBLEM XXII.

To find the solidity of the frustum of an hyperbolic conoid.

RULE.*

Add together the squares of the greatest and least semidiameters, and the square of the whole diameter in the middle, then this sum being multiplied by the altitude, and the product again by .5236 will give the solidity.

---

finitely little, or nothing, the rule will become $\frac{D^2+4\delta^2}{6}\times ap=\overline{D^2+4\delta^2}\times a\times .5236$. Q. E. D.

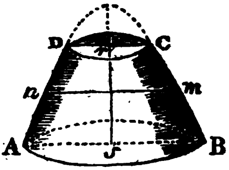
* The demonstration of this rule is contained in that of the last problem.

Or, if D=middle diameter, m=that at $\frac{1}{4}$ of the length, S= generating area of the hyperbola, L=length of the spindle, and $p=3.1416$.

Then will $(D^2+\frac{4m^2-3D^2}{4m-3D}\times\frac{3S}{L}-D)\times\frac{1}{6}pL=$ solidity of the spindle. And if the generating hyperbola be equilateral, then will $(3S\times\frac{L^2+D^2}{LD}-L^2)\times\frac{1}{6}pL=$solidity of the spindle.

EXAMPLES.

1. In the hyperbolic frustum ADCB, the length rs is 20, the diameter AB of the greater end 32, that DC of the less end 24, and the middle diameter nm 28.1708; required the solidity.



Here $(16^2+12^2+28.1708^2)\times20\times.52359=(256+144+793.5939)\times20\times.52359=1193.5939\times20\times.52359=23871.878\times.52359=12499.07660202=$ *solidity required.*

2. Required the solidity of the frustum of an hyperbolic conoid, the height being 12, the greatest diameter 10, the least diameter 6, and the middle diameter $8\frac{1}{2}$. Ans. 667.59.

And, if $l=$ length of the frustum, $s=$ generating area, and the other letters as before; then will $\left(2\text{D}^2+d^2+\frac{4m^2-3\text{D}^2-d^2}{4m-3\text{D}-d}\times\overline{\frac{3s}{l}+d-\text{D}}\right)\times\frac{1}{12}pl=$ solidity of the middle frustum of an hyperbolic spindle.

But if the generating hyperbola be equilateral, the frustum will be $=\left(\frac{3}{2}d^2-l^2+\frac{3s}{l}\times\frac{l^2+\text{D}^2-d^2}{\text{D}-d}\right)\times\frac{1}{6}pl$.

Note.—The content of any spindle formed by the revolution of a conic section about its axis may be found by the following rule:

Add together the squares of the greatest and least diameters, and the square of double the diameter in the middle between the two, and this sum multiplied by the length and the product again by .1309 will give the solidity.

And the rule will never deviate much from the truth when the figure revolves about any other line which is not the axis.

3. What is the content of the middle frustum of an hyperbolic spindle, the length being 20, the middle or greatest diameter 16, the diameter at each end 12, and the diameter at ¼ of the length 14½? Ans. 3248.938.

4. Required the content of the segment of any spindle, its length being 10, the greatest diameter 8, and the middle diameter 6. Ans. 272.272.

Miscellaneous Questions in Solids.

1. If the diameter of the earth be 7930 miles, and that of the moon 2160 miles, required the ratio of their surfaces, and also of their solidities, supposing both of them to be globular, as they are very nearly.

The surfaces of all similar solids are to each other as the squares of their like dimensions; such as diameters, circumferences, like linear sides, &c. &c. And their solidities, as the cubes of those dimensions.

Hence the surface of the moon : *surface of the earth* : : $2160^2 : 7930^2$ *and* * $\frac{2160^2}{7930^2}=\frac{4665600}{62884900}=\frac{1}{13.47}$ *or*, *As* 1 : 13½ *nearly*.

Also the solidity of the moon : *solidity of the earth* : : $2160^3 : 7930^3$ *and* $\frac{2160^3}{7930^3}=\frac{1}{49.5}$ *nearly*, or, *As* 1 : 49½ Ans.

2. Three persons having bought a sugar loaf, want to divide it equally among them by sections parallel to the base; it is required to find the altitude of each person's share, supposing the loaf to be a cone whose height is 20 inches.

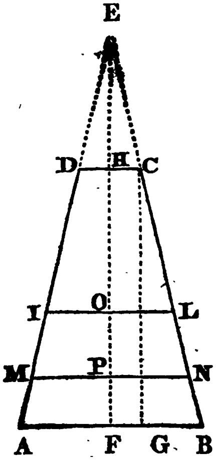
---

* The ratio of one quantity to another may be expressed by dividing the antecedent by the consequent.

By the similar cones we have[*] $3 : 1 :: 20^3 : \frac{20^3}{3} =$ *the cube of the height of the top sections; wherefore* $\sqrt[3]{\frac{20^3}{3}} =$ 13.867 *the upper part.*

Also $3 : 2 :: 20^3 : \frac{2\times 20^3}{3}$ *and* $\sqrt[3]{\frac{2\times 20^3}{3}} - 13.867 =$ 3.604 *the middle part; wherefore the lower part will be* 2.528.

3. Three men bought a tapering piece of timber, which was the frustum of a square pyramid; one side of the greater end was 3 feet, one side of the less end 1 foot, and the length 18 feet; what is the length of each man's piece, supposing they paid equally, and are to have equal shares?



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[*] This proportion, as well as all others of the kind, may be expressed thus: $\sqrt[3]{3} : \sqrt[3]{1} :: 20 : \frac{20}{\sqrt[3]{3}} =$ the height of the top section; and, in some instances, this is the more convenient method.

Let ABCDE *be a section of the pyramid (when completed) passing through the vertex, and bisecting the opposite sides of the base, and let* IL *and* MN *represent the required sections. Draw* EF *to the middle of* AB *and draw* CG *parallel to it.*

Then by similar triangles BG (1 *foot*) : GC(18) :: BF (1.5) : FE(27) *and* FE—FH=EH=9, *the altitude of the pyramid* EDC.

Hence Prob. VIII. of solids the solidities of the two pyramids EAB *and* EDC *will be found* 81 *and* 3 *cubic feet respectively, and* 81—3=78= *the solidity of the frustum* ABCD. *Also* $\frac{78}{3}=26$, *the solidity of each person's share, which added to the solidity of* EDC, *will give the solidity of* EIL=29, *and the double of it added to* EDC *will give the solidity of* EMN=55.

Now in the similar pyramids, EDC(3) : EIL(29) :: $EH^3 : EO^3$, *the cube root of which will give* EO *and* EO—EH =HO *the length of the part adjacent to the less end*= 10.172.

Again EIL(29) : EMN(55) :: $EO^3 : EP^3$ *the cube root of which will give* EP *and* EP—EO=OP *the length of the middle part*=4.559. *Lastly*, EF—EP=PF *the length of the part adjacent to the larger end*=3.269.

4. If a round pillar, 7 inches over, have 4 feet of stone in it; of what diameter is the column, of equal length, that contains 10 times as much?

The solidities of cylinders, prisms, parallelopipedons, &c. which have their altitudes equal, are to each other as the squares of their diameters or like sides. The same remark is applicable to frustums of a cone or pyramid when the altitude is the same, and the ends proportional.

Hence, As 4 : 40, *or As* 1 : 10 :: 7^2 : 490= *the square of the required diameter, and* $\sqrt{490}=22.1359$ *the diameter required.*

5. There is a mill-hopper, in the form of a square pyramid, whose solid content is 13½ feet; but one foot is cut off

its perpendicular altitude to make a passage for the grain, from the frustum or hopper to the mill-stone: the sides of its greater and less end are in proportion of $4\frac{1}{2}$ to 1. Required the content in dry or corn measure.

Ans. 10.7292 bushels.

6. The ditch of a fortification is 1000 feet long, 9 feet deep, 20 feet broad at bottom, and 22 at top; how much water will fill the ditch, allowing 282 cubic inches to make a gallon? Ans. $1158127\frac{31}{47}$ gallons.

7. A person having a frustum of a cone 12 inches in height, and the diameters of the greater and smaller ends 5 and 3 inches respectively, wishes to know the diameter of a frustum of the same altitude, that would contain 3666 cubic inches, and have its diameters in the same proportion as the smaller one.

Ans. *The greater diameter* 24.4002, *and less* 14.6401.

OF THE

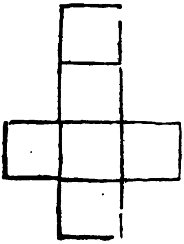
REGULAR BODIES.

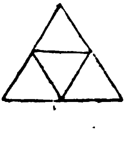
A REGULAR BODY is a solid contained under a certain number of similar and equal plane figures.

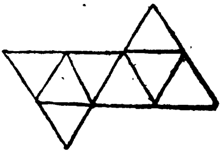
The whole number of regular bodies which can possibly be formed is five.

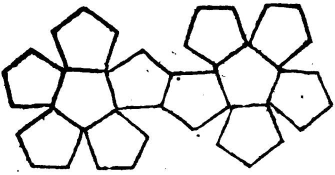
1. The *Tetraedron*, or regular pyramid, which has four triangular faces.
2. The *Hexaedron*, or cube, which has six square faces.
3. The *Octaedron*, which has eight triangular faces.
4. The *Dodecaedron*, which has twelve pentagonal faces.
5. The *Icosaedron*, which has twenty triangular faces.

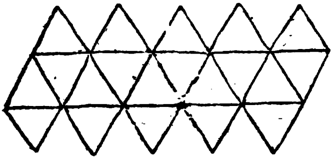
If the following figures are made of pasteboard, and the lines be cut half through, so that the parts may be turned up and glued together, they will represent the five regular bodies here mentioned.











PROBLEM I.

To find the solidity of a tetraedron.

RULE.*

Multiply $\frac{1}{12}$ of the cube of the linear side by the square root of 2, and the product will be the solidity.

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* *Demon.* From one angle C of the tetraedron ABCn, let fall the perpendicular ce, upon the opposite side, and draw Ae.

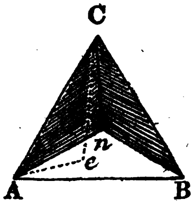
Then $AC^2—Ae^2=ce^2$; and since the point e is equally distant from the three angles A, B, and n, $\frac{1}{3}\ AC^2=(\frac{1}{3}AB^2)$ Ae^2 as is shown in the demonstration of the rule for regular polygons, page 61. Consequently $AC^2—\frac{1}{3}AC^2=\frac{2}{3}AC^2$ $=ce^2$, or $ce=AC\sqrt{\frac{2}{3}}$. But the area of the triangle $AnB=\frac{1}{4}AB^2\sqrt{3}=\frac{1}{4}\ AC^2\sqrt{3}$; and therefore $\frac{1}{2}\ AC\ \sqrt{\frac{2}{3}}\ (\frac{1}{3}\ ce)$ $\times\frac{1}{4}AC^2\sqrt{3}(\triangle AnB)=\frac{1}{12}AC^3\sqrt{2}$. Q. E. D.

If L be put = length of the linear edge, then will $L^2\sqrt{3}$ = whole *surface of the tetraedron.*

The rule for the hexaedron, or cube, has been given before.

EXAMPLES.

1. The linear side of a tetraedron ABC*n* is 4 : what is the solidity ?



$\frac{4^3}{12}\times\sqrt{2}=\frac{4\times4\times4}{12}\times\sqrt{2}=\frac{4\times4}{3}\times\sqrt{2}=\frac{16}{3}\times\sqrt{2}=\frac{16}{3}\times 1.414=\frac{22.624}{3}=7.5413=$ *solidity required.*

2. Required the solidity of a tetraedron whose side is 6.
Ans. 25.452.

PROBLEM II.

To find the solidity of an octaedron.

RULE.*

Multiply ⅓ of the cube of the linear side by the square root of 2, and the product will be the solidity.

---

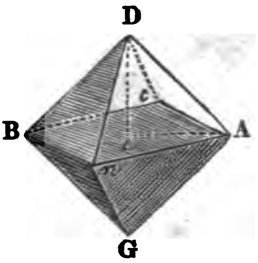
* *Demon.* From the angle D of the octaedron DBGA let fall the perpendicular D*e*.

Then since the solid is composed of two equal square pyramids, each of whose bases B*n*AC are equal to the square of the linear side AG or AD, we shall have $\mathrm{B}n\mathrm{AC}\times\frac{2}{3}\,\mathrm{D}e=\mathrm{A}n^2\times\frac{2}{3}\mathrm{D}e=$ content of the solid.

But D*e* evidently bisects the diagonal BA, and is equal to

EXAMPLES.

1. What is the solidity of the octaedron BGAD, whose linear side is 4?



$\frac{4^3}{3}\times\sqrt{2}=\frac{64}{3}\times\sqrt{2}=21.333$, &c. $\times\sqrt{2}=21.333$, &c. $\times 1.414$, &c. $=30.16486=$ *solidity required.*

2. Required the solidity of an octaedron whose side is 8. Ans. 241.3568.

PROBLEM III.

To find the solidity of a dodecaedron.

RULE.*

To 21 times the square root of 5 add 47, and divide the sum by 40: then the square root of the quotient being

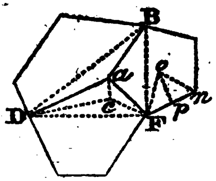
---

Be; therefore $An^2\times\frac{2}{3}De=An^2\times\frac{2}{3}Be=An^2\times\frac{1}{3}BA=\frac{1}{3}An^2\times\sqrt{An^2+Ac^2}=\frac{1}{3}An^2\sqrt{2An^2}=\frac{1}{3}An^3\sqrt{2}$. Q. E. D.

If $L=$ linear side as before, then will $2L^2\sqrt{3}=$ surface of the octaedron.

[*] *Demon.* Let *a* be a solid angle of the dodecaedron, and *ac* a

multiplied by 5 times the cube of the linear side will give the solidity required.



perpendicular falling on the equilateral plane BDF. Also join the points D, c and c, F.

Then the angle DaF contains 108 degrees, whose sine is $\frac{1}{4}\sqrt{10+2\sqrt{5}}$, and the angle aFD contains 36 degrees, whose sine is $\frac{1}{4}\sqrt{10-2\sqrt{5}}$, the radius in both cases being taken equal to 1.

Therefore, by Trigonometry, $\frac{1}{4}\sqrt{10-2\sqrt{5}} : \frac{1}{4}\sqrt{10+2\sqrt{5}}$

$$:: aD : DF = aD\sqrt{\frac{5+\sqrt{5}}{5-\sqrt{5}}} = aD\frac{1+\sqrt{5}}{2} = \tfrac{1}{2}+\tfrac{1}{2}\sqrt{5}\times aD.$$

Again, since c is the centre of DBF, the angles cDF and cFD are each 30°, and the angle DcF$=120°$; but the sine of 30° is $\frac{1}{2}$; and the sine of 120° is $\frac{1}{2}\sqrt{3}$; whence, by trigonometry, $\frac{1}{2}\sqrt{3}$

$$: DF :: \tfrac{1}{2} : DC = \frac{DF}{\sqrt{3}} = aD\frac{1+\sqrt{5}}{2\sqrt{3}};$$ and consequently $ac=$

$$\sqrt{aD^2 - DC^2} = aD\sqrt{\frac{3-\sqrt{5}}{6}} = aD\sqrt{\tfrac{1}{2}-\tfrac{1}{6}\sqrt{5}}.$$

But a perpendicular from a upon the plane BDF must pass through the centre of the circumscribing sphere, and ac will be the versed sine of an arc whose chord is aD, and radius equal to that of the said sphere.

Whence $ac : aD :: a : \frac{aD^2}{ac} = \frac{aD^2}{aD\sqrt{\frac{1}{2}-\frac{1}{6}\sqrt{5}}} = aD\,\frac{\sqrt{3}+\sqrt{15}}{2}$

$=$diameter of the circumscribing sphere, and $aD\,\frac{\sqrt{3}+\sqrt{15}}{4}$

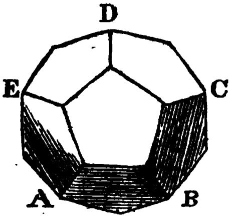
$=R=$radius of the circumscribing sphere.

Again, the angle Fon contains 72°, whose sine is $\frac{1}{4}\sqrt{10+2\sqrt{5}}$; and the angle oFn is 54°, whose sine is $\frac{1+\sqrt{5}}{4}$; whence by trigonometry, $\frac{1}{4}\sqrt{10+2\sqrt{5}} : \frac{1+\sqrt{5}}{4} :: Fn\ (aD) : oF = aD$

$$\frac{1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}} = aD\sqrt{\frac{5+\sqrt{5}}{10}} = aD\sqrt{\tfrac{1}{2}+\tfrac{1}{10}\sqrt{5}}.$$

EXAMPLES.

1. The linear side of the dodecaedron ABCDE is 3; what is the solidity?



$\sqrt{\frac{21\sqrt{5}+47}{40}}\times 27\times 5=\sqrt{\frac{21\times 2.23606+47}{40}}\times 27\times 5=$
$\sqrt{\frac{46.95726+47}{40}}\times 135=206.901$ *solidity required.*

2. The linear side of a dodecaedron is 1; what is the solidity? Ans. 7.6631.

But since the radius of the circumscribing sphere is the hypothenuse of a right angled triangle, whose legs are oF and the radius of the inscribed sphere, we shall have $\sqrt{R^2-oF^2}=$
$\sqrt{(\frac{1}{4}\sqrt{3}+\sqrt{15}aD)^2-\frac{1}{2}+\frac{1}{10}\sqrt{5}aD^2}=aD\sqrt{\frac{25+11\sqrt{5}}{40}}=$
$\sqrt{\frac{5}{8}+\frac{11}{40}\sqrt{5}}$=radius of the inscribed sphere.

And because the solid is composed of 12 equal pentagonal pyramids, each of whose bases are by Prob. VIII.$=\frac{5aD^2}{4}\sqrt{1+\frac{2}{5}\sqrt{5}}$;
therefore $\frac{60aD^2}{4}\sqrt{1+\frac{2}{5}\sqrt{5}}\times\frac{1}{3}r=\frac{60aD^2}{4}\sqrt{1+\frac{2}{5}\sqrt{5}}\times\frac{aD}{3}$
$\sqrt{\frac{25+11\sqrt{5}}{40}}=5aD^3\sqrt{\frac{47+21\sqrt{5}}{40}}$=solidity of the dodecaedron
Q. E. D.

If L be put for the linear side, then will $15L^2\sqrt{\frac{5+2\sqrt{5}}{5}}=$ surface of the dodecaedron.

PROBLEM IV.

To find the solidity of an icosaedron.

RULE.*

To 3 times the square root of 5 add 7, and divide the sum by 2; then the square root of this quotient being multiplied by $\frac{5}{6}$ of the cube of the linear side will give the solidity required.

That is $\frac{5}{6}S^3 \times \surd\left(\frac{7+3\surd 5}{2}\right)=$ solidity when S is $=$ to the linear side.

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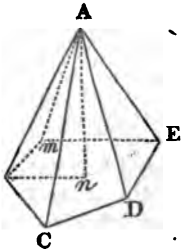
* *Demon.* Let A be a solid angle of the icosaedron, formed by 5 faces, or triangles, whose bases make the pentagon BCDEm.



Then, if a perpendicular be demitted from A upon the pentagonal plane BCDEm, it will fall into the centre n, and Bn, by the demonstration of the last problem, will be $=\text{AB}\surd\frac{5+\surd 5}{10}$ and the radius of the circle circumscribing one of the faces ABC$=\frac{1}{3}\text{AB}\surd 3$.

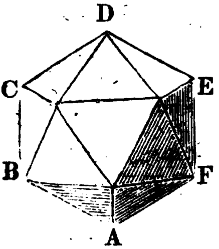
But the radius of the circumscribing sphere is $\text{R}=\frac{\text{BA}^2}{2\text{A}n}=\frac{\text{BA}^2}{2\surd\overline{\text{AB}^2-\text{B}n^2}}=\text{AB}\surd\frac{5+\surd 5}{8}$, found as in the last problem.

And, since R is the hypothenuse of a right angled triangle, one of whose legs is $\frac{1}{3}\text{AB}\surd 3$, the radius of the circle circumscribing the face ABC, and the other r, the radius of the inscribed sphere, we shall have $r=\surd\overline{\text{R}^2-\frac{1}{3}(\text{AB}\surd 3)^2}=\surd\overline{\frac{5+\surd 5}{8}\text{AB}^2-\frac{1}{3}\text{AB}^2}=\text{AB}\surd\frac{7+3\surd 5}{24}$.

But the solid is composed of 20 equal triangular pyramids, each of whose bases is $=\frac{\text{AB}^2}{4}\surd 3$ by Problem VIII; therefore $\frac{20\text{AB}^3}{4}$

EXAMPLES.

1. The linear side of the icosaedron ABCDEF is 3; what is the solidity?



$$\sqrt{\frac{3\sqrt{5}+7}{2}} \times \frac{5\times 3^3}{6} = \sqrt{\frac{3\times 2.23606+7}{2}} \times \frac{5\times 27}{6} =$$

$$\sqrt{3}\times\tfrac{1}{3}r = 5AB^2\sqrt{3}\times\frac{AB}{3}\sqrt{\frac{7+3\sqrt{5}}{24}} = \tfrac{3}{6}\,AB^3\sqrt{\frac{7+3\sqrt{5}}{2}}$$

=solidity of the icosaedron. Q. E. D.

If L be put for the linear side, then will $5L^2\sqrt{3}$=surface of the icosaedron.

Note.—The superficies and solidity of any of the five regular bodies may be found as follows.

Rule 1. Multiply the tabular area by the square of the linear edge, and the product will be the superficies.

2. Multiply the tabular solidity by the cube of the linear edge, and the product will be the solidity.

Surfaces and Solidities of the Regular Bodies.

No. of sides.	Names.	Surfaces.	Solidities.
4	Tetraedron	1.73205	0.11785
6	Hexaedron	6.00000	1.00000
8	Octaedron	3.46410	0.47140
12	Dodecaedron	20.64578	7.66312
20	Icosaedron	8.66025	2.18169

$\sqrt{\frac{6.70818+7}{2}} \times \frac{5\times 9}{2} = \sqrt{\frac{13.70818}{2}} \times \frac{45}{2} = \sqrt{6.85409} \times$ $22.5 = 2.61803 \times 22.5 = 58.9056 =$ *solidity required.*

2. Required the solidity of an icosaedron, whose linear side is 1? Ans. 2.1820.

OF

CYLINDRIC RINGS.

PROBLEM I.

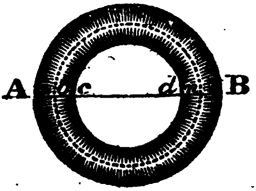
To find the convex superficies of a cylindric ring.

RULE.*

To the thickness of the ring add the inner diameter, and this sum being multiplied by the thickness, and the product again by 9.8696 will give the superficies required.

EXAMPLES.

1. The thickness of *Ac* of a cylindric ring is 3 inches, and the inner diameter *cd* 12 inches; what is the convex superficies?



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* A solid ring of this kind is only a bent cylinder, and therefore the rules for obtaining its superficies, or solidity, are the same as those already given. For let A*c* be any section of the solid perpendicular to its axis *on*, and then A*c*×3.14159, &c.=circumference of that section, and $\overline{Ac+cd}$ (*on*)×3.14159, &c.=length of the axis *on*.

$\overline{12+3} \times 3 \times 9.8696 = 15 \times 3 \times 9.8696 = 45 \times 9.8696 =$ $444.132 =$ *superficies required.*

2. The thickness of a cylindric ring is 4 inches, and the inner diameter 18; what is the convex superficies?

Ans. 868.52 square inches.

3. The thickness of a cylindric ring is 2 inches, and the inner diameter 18; what is the convex superficies?

Ans. 394.784 square inches.

PROBLEM II.

To find the solidity of a cylindric ring.

RULE.*

To the thickness of the ring add the inner diameter, and this sum being multiplied by the square of half the thickness, and the product again by 9.8696 will give the solidity.

Whence $Ac \times 3.14159$, &c. $\times \overline{Ac+cd} \times 3.14159$, &c. $=$ $\overline{Ac+cd} \times Ac \times \overline{3.14159,\ \&c.}\,|^2 = \overline{Ac+cd} \times Ac \times 9.8696$, &c. $=$superficies required.

* *Demon.* $Ac^2 \times .78539$, &c. $= Ac^2 \times \frac{3.14159}{4}$, &c. $= \frac{1}{4}$ $Ac^2 \times 3.14159$, &c. $=$area of the section Ac; and $\overline{Ac+cd}$ $(on) \times 3.14159$, &c. $=$length of the axis *on*.

Therefore $\overline{Ac+cd} \times \overline{\frac{1}{4}Ac^2 \times 3.14159,\ \&c.}\,|^2 = ac+cd \times \frac{1}{4}$ $Ac^2 \times 9.8696$. Q. E. D.

This figure being only a cylinder bent round into a ring, its surface and solidity may also be found as in the cylinder, namely, by multiplying the axis or length of the cylinder by the circumference of the ring, or section, for the surface, and by the area of a section for the solidity.

Thus, if $c=$circumference of the ring, or section, $a=$area of that section, and $l=$length of the axis; then will $cl=$surface of the ring, and $al=$to its solidity. Which rules are the same as for the cylinder, and may be easily converted into those given in the text.

These rules are indeed so obvious, as to render any demonstration of them altogether unnecessary.

EXAMPLES.

1. What is the solidity of an anchor ring, whose inner diameter is 8 inches, and thickness in metal 3 inches ?

$\overline{8+3} \times \left.\frac{3}{2}\right]^2 \times 9.8696 = 11 \times 1.5^2 \times 9.8696 = 11 \times 2.25 \times 9.8696 = 24.75 \times 9.8696 = 244.2726 =$ *solidity required.*

2. The inner diameter of a cylindric ring is 18 inches and its thickness 4 inches ; what is the solidity ?

Ans. 868.5248.

3. Required the solidity of a cylindric ring, whose thickness is 2 inches, and its inner diameter 12,

Ans. 138.1744.

4. What is the solidity of a cylindric ring, whose thickness is 4 inches, and inner diameter 16 ? Ans. 789.568.

OF

ARTIFICERS' WORK.

Artificers estimate or compute the value of their works by different measures, viz.*

1. *Glazing* and *Mason's* flat work, &c. by the foot.
2. *Painting*, *Plastering*, *Paving*, &c. by the yard.
3. *Flooring*, *Partitioning*, *Roofing*, *Tiling*, &c. by the square of 100 feet.
4. *Brickwork*, &c. by the rod of 16½ feet, whose square is 272¼.

The measures made use of in these works are contained in the following table:

12 inches	make	1 lineal foot
144 square inches		1 square foot
9 square feet		1 square yard
100 square feet		a square
272¼ square feet or, 30¼ square yards,		1 rod, perch, or square pole.

---

* The best method of taking the dimensions of all sorts of artificers' work is by feet, tenths, and hundredths; because the computation may then be performed by common multiplication, or by the sliding rule, hereafter described.

OF

BRICKLAYERS' WORK.*

BRICKLAYERS compute or value their work at the rate of a brick and a half thick, and, if the wall be more or less than this standard, it must be reduced to it as follows:

---

* *Note.*—In practice it is usual to divide the square feet by 272 only, omitting the ¼; but the more accurate way is, to divide by 272.25.

The usual way to take the dimensions of a building is to measure half round its middle, on the outside, and half round it on the inside; and this will give the true compass, in which the thickness of the wall is included.

When the height of the building is unequal, take several different altitudes, and their sum, being divided by the number of altitudes you have taken, may be considered as the mean height.

To measure a chimney standing by itself, without any party-wall adjoining; girt it about for the length, and reckon the height of the story for the breadth; but if it stands against a wall, you must measure it round to the wall for the girt, and take the height as before.

When the chimney is wrought upright from the mantel-tree to the ceiling, the thickness must always be the same with the jambs; and nothing is ever deducted for the vacancy between the floor and the mantel-tree, because of the gathering of the breast and wings to make room for the hearth in the next story.

To measure chimney shafts, or that part which appears above the roof; girt them with a line, about the least place for the length, and take the height for the breadth; and if they be four inches thick, set down the thickness at one brickwork; but if they are nine inches thick, reckon it a brick and a half, in consideration of the plastering and scaffolding.

All windows, doors, &c. are to be deducted out of the contents of the walls in which they are placed. But this deduetion is made only with regard to materials; for the value of their workmanship is added to the bill at the stated rate agreed on,

RULE.

Multiply the superficial content of the wall in feet, by the number of half bricks in the thickness, and ⅓ of that product will be the content required.

Note.—In America bricklayers' work is generally reckoned by the 1000.

EXAMPLES.

1.* How many square rods are there in a wall 52½ feet long, 12 feet 9 inches high, and 2½ bricks thick?

By Decimals.

$$\text{Here}\ \frac{52.5\times12.75}{272}=\frac{669.375}{272}=2.4609$$

$$\begin{array}{r} 2.4609 \\ 5\ \textit{half bricks.} \\ \hline 3)12.3045 \\ \hline 4.1015 = 4\ 27\ 7\ \textit{answer.} \end{array}$$

ro. ft. in.

There are also other allowances to be made to the workmen, such as those for returns or angles made by two adjoining walls, and double measure for feathered gable ends, &c.

All ornamental work is generally valued by the foot square, such as arches, doors, architraves, friezes, cornices, &c. But carved mouldings, &c. are often agreed for by the running foot, or lineal measure.

* In this and the following examples, 272 feet are used for a rod.

By Cross Multiplication.

```
      feet in.
       52  6
       12  9
      ───────
      630  0
       39  4  6
      ──────────
272)669  4  6
      ──────────
       2 124  4
              5
      ──────────
   3)12  82  8
      ──────────
       4  27  7 as before.
```

2. How many square rods are there in a wall 62½ feet long, 14 feet 8 inches high, and 2½ bricks thick?

ro. fe. in. p.
Ans. 5 167 9 4

3. If each side wall of a building be 45 feet long on the outside, each end wall 15 feet broad on the inside, the height of the building 20 feet, and the gable at each end of the wall 6 feet high, the whole being 2 bricks thick; what is the true content in standard rods?

Ans. 12.2059.

OF

MASONS' WORK.

To *Masonry* belong all sorts of stone work, and the measure made use of is a solid perch, or a superficial or solid foot.*

EXAMPLES.

1. Required the solid content of a wall whose length is 48 feet 6 inches, its height 10 feet 9 inches, and thickness 2 feet.

By Decimals.

$48.5 \times 10.75 \times 2 = 1042.75$ *feet*. Ans.

---

* Solid measure is principally used for materials, and the superficial for workmanship.—In the solid measure, the true length, breadth, and thickness, are taken, and multiplied continually together. And in the superficial measure, the length and breadth of every part of the projection must be taken, as it appears without the general upright face of the building.

Masons, in measuring their work, usually take the whole girt of the building, that is, the length of a string that passes entirely round the building, which is 4 times the thickness of the wall more than the true measure. This is added on account of the trouble of carrying up the corners.

In America, the thickness of the wall is not reckoned to the mason at less than 18 inches; but if it is more than that thickness it is reduced to it. No deduction of the mason work is made for doors, windows, &c. on account of the trouble of carrying up the straight walls on the sides of them.

All windows, doors, &c. are to be deducted out of the contents of the walls in which they are placed with regard to materials.

By Cross Multiplication.

feet	in.	
48	6	
10	9	
485	0	
36	4	6
521	4	6
2	0	0
1042	9	0 *the same as before.*

2. Required the solid content of a wall whose length is 53 feet 6 inches, its height 12 feet 3 inches, and its thickness 2 feet. Ans. 1310 feet 9 in.

3. What is a marble slab worth, whose length is 5 feet 7 inches, and breadth 1 foot 10 inches, at 80 cents per foot?

dolls. cts. m.
Ans. 8 18 8.

4. What is the solid content of a wall, whose length is 60 feet 9 inches, its height 10 feet 3 inches, and its thickness 2½ feet? Ans. 1556.71875 feet.

5. In a chimney-piece, suppose the

	fe.	in.
Length of the mantel and slab, each - -	4	6
Breadth of both together, - - - -	3	2
Length of each jamb, - - - - - -	4	4
Breadth of both together, - - - -	1	9

What will be the content of the chimney-piece?

Ans. 21 feet 10 in.

6. The dimensions of a certain building are as follow: viz. 58 feet by 26 on the outside; height of the building 22 feet, height of the gable at each end 12 feet, thickness of the wall 15 inches. Two doors 4½ by 8 feet each, 28 windows each 3½ by 6 feet: what will the mason work amount to at 56 cents a perch, and what must be paid for the stone at 44 cents a perch?

Ans. *For the work,* $136 03; *For the stone,* $71 95½.

OF

CARPENTERS' AND JOINERS' WORK.*

Carpenters' and Joiners' work is that of flooring, partitioning, roofing, &c. and is measured by the square of 100 feet.

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*Note.—Large and plain articles are usually measured by the foot, or yard, &c. square; but enriched mouldings, and some other articles, are often estimated by running or lineal measure, and some things are rated by the piece.

In measuring of joists it is to be observed, that only one of their dimensions is the same with that of the floor, and the other will exceed the length of the room by the thickness of the wall, and one-third of the same, because each end is let into the wall about two-thirds of its thickness.

No deductions are made for hearths, on account of the additional trouble and waste of materials.

Partitions are measured from wall to wall for one dimension, and from floor to floor, as far as they extend, for the other.

No deduction is made for door ways, on account of the trouble of framing them.

In measuring of Joiners' work, the string is made to ply close to every part of the work over which it passes.

In roofing, the length of the house in the inside, together with two-thirds of the thickness of one gable, is to be considered as the length, and the breadth is equal to double the length of a string, which is stretched from the ridge down to the rafter, along the eaves board, till it meets with the top of the wall.

For stair-cases, take the breadth of all the steps, and make a line ply close over them, from the top to the bottom, and multiply the length of this line by the length of a step for the whole area. By the length of a step is meant the length of the front and the returns at the two ends, and by the breadth is to be understood the girt of its two upper surfaces, or the tread and riser.

EXAMPLES.

1. If a floor be 57 feet 3 inches long, and 28 feet 6 inches broad, how many squares will it contain ?

By Decimals.

$57.25 \times 28.5 = 1631.625$ *sq. ft.* = 16 *sq.* 31 *ft.* 7 *in.* 6″. Ans.

By Cross Multiplication.

feet	*in.*	
57	3	
28	6	
1603	0	
28	7	6
16|31	7	6

For the balustrade, take the whole length of the upper part of the hand-rail, and girt over its end till it meet the top of the newel-post, for the length; and twice the length of the baluster upon the landing, with the girt of the hand-rail, for the breadth.

For wainscotting, take the compass of the room for the length, and the height from the floor to the ceiling, making the string ply close into all the mouldings, from the breadth. Out of this must be made deductions for windows, doors, and chimneys, &c. but workmanship is counted for the whole, on account of the extraordinary trouble.

For doors, it is usual to allow for their thickness, by adding it into both the dimensions of length and breadth, and then multiplying them together for the area. If the door be pannelled on both sides, take double its measure for the workmanship; but if one side only be pannelled, take the area and its half for the workmanship.

For the surrounding architrave, gird it about the outermost part for its length; and measure over it as far as it can be seen when the door is open, for the breadth.

Window-shutters, *bases*, &c. are measured in the same manner.

In the measuring of roofing for workmanship alone, all holes for chimneys-shafts and sky-lights are generally deducted.

But in measuring for work and materials, they commonly measure in all sky-lights, luthern-lights, and holes for the chimney-shafts, on account of their trouble and waste of materials.

2. A floor is 53 feet 6 inches long, and 47 feet 9 inches broad: how many squares does it contain?

Ans. 25 sq. and 54 feet 7½ in.

3. A partition is 91 feet 9 inches long, and 11 feet 3 inches broad: how many squares does it contain?

Ans. 10 sq. and 32 feet.

4. *If a house within the walls be 44 feet 6 inches long, and 18 feet 3 inches broad; how many squares of roofing will cover it, allowing the roof to be of a true pitch?

Ans. 12 sq. and 18 feet.

5. If a house measure within the walls 52 feet 8 inches in length, and 30 feet 6 inches in breadth, and the roof be of a true pitch, what will it cost roofing, at 1 dollar 40 cts. per square?

Ans. 33 dolls. 73 cts. 3 m.

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* It is customary to reckon the flat and half of any building within the walls, for the measure of the roof of that building, when the roof is of a *true pitch*, or so that the length of the rafters is ¾ of the breadth of the building.

OF

SLATERS' AND TILERS' WORK.

In these works the content of a roof is found by multiplying the length of a ridge by the girt from eave to eave; and, in slating, allowance must be made for the double row at the bottom.

In taking the girt, the line is made to ply over the lowest row of slates, and returned up the under side till it meet with the wall or eaves-board; but in tiling, the line is stretched down only to the lowest part, without returning it up again.

Double measure is generally allowed for hips, valleys, gutters, &c. but no deductions are made for chimneys.*

---

* In angles formed in a roof, running from the ridge to the eaves, that angle of the roof which bends inwards, is called a *valley:* and the angle bending outwards is called a *hip*. And in tiling and slating, it is common to add the length of the valleys to the content in feet; and sometimes also the hips are added.

In slating it is common to reckon the breadth of the roof 2 or 3 inches broader than what it measures, because the first row is almost covered by the second; and this is done sometimes when a roof is tiled.

Note.—Sky-lights and chimney-shafts are always deducted; but they seldom deduct luthern-lights, or garret-windows on the roof; for the covering them is reckoned equal to the hole in the roof.

In all works of this kind the content is computed, either in yards of 9 square feet, or in squares of a hundred feet, and the same allowance of hips and valleys is to be made as in roofing.

EXAMPLES.

1. The length of a slated roof is 45 feet 9 inches, and its girt 34 feet 3 inches; what is the content?

By Decimals.

$45.75 \times 34.25 = 1566.9375$ *sq. ft.* $= 174.104$ *yds.* Ans

By Cross Multiplication.

feet	*in.*		
45	9		
34	3		
1555	6		
11	5	3	
9)1566	11	3	
174	0	11	3

Ans. 174 yds.

2. What will the tiling of a barn cost at $3 40 per square, the length being 43 feet 10 inches, and the breadth 27 feet 5 inches, on the flat, the eave boards projecting 16 inches on each side, allowing the roof to be of a true pitch?
Ans. $65 26

PLASTERERS' WORK.

PLASTERERS' work is of two kinds, viz. plastering upon laths, called ceiling, and plastering upon walls, called rendering; and these different kinds must be measured separately, and their contents collected into one sum.*

Note.—Proper deductions must be made for doors, windows, &c.

EXAMPLES.

If a ceiling be 59 feet 9 inches long, and 24 feet 6 inches broad; how many yards does it contain?

By Decimals.

59.75 × 24.5 = 1463.875 *sq. feet* = 162.6528 *sq. yds.* Ans.

By Cross Multiplication.

	feet	*in.*	
	59	9	
	24	6	
	1434	0	
	29	10	6
9)	1463	10	6
	162 5	10	6

Ans. 162 yards 5 feet.

---

* Plasterers' plain work is measured by the square foot, or yard of 9 square feet, and enriched mouldings, &c. by running or lineal measure.

2. If the partitions between rooms are 141 feet 6 inches about, and 11 feet 3 inches high; how many yards do they contain? Ans. 176.87.

3. The length of a room is 14 feet 5 inches, its breadth 13 feet 2 inches, and height 9 feet 3 inches to the under side of the cornice, whose girt is 8½ inches, and its projection 5 inches from the wall on the upper part next the ceiling; what will be the quantity of plastering, supposing there be no deductions but for one door, whose size is 7 feet by 4 feet?

Ans. 53 *yds.* 5 *ft.* 3 *in. of rendering.*
18 *yds.* 5 *ft.* 6 *in. of ceiling.*
And 39 *ft.* $0\frac{1}{12}$ *in. of cornice.*

OF

PAINTERS' AND GLAZIERS' WORK.

PAINTERS' work is measured in the same manner as that of Carpenters; and in taking the dimensions the line must be forced into all the mouldings and corners.

The work is estimated at so much per yard, except sashes, which are done at so much per light; in carved mouldings, &c. it is customary to allow double the usual measure.

Glaziers' work is done at so much per light, that is, computed at a given price for putting in each pane of glass, according to the size.

EXAMPLES.

1. If a room be painted, whose height is 16 feet 6 inches, and its compass 97 feet 9 inches; how many yards does it contain?

Balustrades and most other works of that kind are measured as in Joiners' work, excepting for doors, window-shutters, &c. where the Joiner is only allowed the area and half area; but the Painter has always double the area of one side, because every part that is painted must be measured.

Note.—Painters take their dimensions with a string, and measure from the top of the cornice to the floor, girting the string over all the mouldings and swellings; and their price is generally proportioned to the number of times they lay on their colour.

All work of this kind is done by the square yard, and every part where the colour lies must be measured and estimated in the general account of the work.

Deductions are to be made for chimneys, casements, &c. and the price is generally proportioned to the number of times they lay on their colour.

By Decimals.

97.75 × 16.5 = 1612.875 sq. feet = 179.208 sq. yds. Answer.

By Cross Multiplication.

97	9	
16	6	
1564	0	
48	10	6
9)1612	10	6
179 1	10	6

Ans. 179 yards 1 foot.

2. The height of a room is 14 feet 10 inches, and the circumference 21 feet 8 inches, how many square yards does it contain? Ans. 35.71.

3. Suppose a room, that was to be painted at 8d. per yard, measures as follows: the height is 11 feet 7 inches; the girt or compass 74 feet 10 inches; the door 7 feet 6 inches, by 3 feet 9 inches; five window-shutters, each 6 feet 8 inches, by 3 feet 4 inches; the breaks in the windows 14 inches deep and 8 feet high; the chimney 6 feet 9 inches by 5 feet; a closet, the height of the room, 3½ feet deep, and 4¾ feet in front, with shelving, at 22 feet 6 inches by 10 inches; the shutters, doors and shelves being all coloured on both sides; what will the whole come to? Ans. 4*l.* 18s. 9d.

OF

PAVIOURS' WORK.

Paviours' work is done by the square yard, and the content is found by multiplying the length by the breadth.

Or if the dimensions be taken in feet, and the area be found in the same measure, the result being divided by 9, will give the number of square yards required.

EXAMPLES.

1. What will the paving of a rectangular court-yard come to at 3s. 2d. per yard, supposing the length to be 27 feet 10 inches, and the breadth 14 feet 9 inches?

Plumbers' work is generally done by the pound, or hundred weight, and the price is regulated according to the value of the lead at the time the contract is made, or when the work is performed.

Sheet lead, used in roofing, guttering, &c. is generally between 7 and 12lbs. weight to the square foot.

The following table will show the weight of a square foot to each of these thicknesses.

Thick.	*lbs. sq. foot.*	*Thick.*	*lbs. sq. foot.*	*Thick.*	*lbs. sq. foot.*
$\frac{1}{8}$	7.373	.15	8.848	.18	10.618
.13	7.668	.16	9.438	.19	11.207
.14	8.258	$\frac{1}{6}$	9.831	$\frac{1}{5}$	11.797
$\frac{1}{7}$	8.427	.17	10.028	.21	12.387

By Cross Multiplication.

feet	*in.*			
27	10			
14	9			
389	8			
20	10	6		
9)410	6	6		
45	5	6	6	*at* 3*s.* 2*d.*

s.	*d.*					
2	0		$\frac{1}{10}$	45		
1	0		$\frac{1}{2}$	4	10	
	2		$\frac{1}{6}$	2	5	
f.	*in.*	*p.*				
3	0	0	$\frac{1}{3}$		7	6
1	0	0	$\frac{1}{3}$		1	0½
1	0	0	$\frac{1}{1}$		0	4
0	6	0	$\frac{1}{2}$		0	4
0	0	6	$\frac{1}{12}$		0	2
					0	0
				7	4	4½ the answer.

2. A rectangular court yard is 42 feet 9 inches long, and 68 feet 6 inches in depth, and a foot-way goes quite through it, of 5 feet 6 inches in breadth; the foot-way is laid with stone at 3*s.* 6*d.* per yard, and the rest with pebbles at 3*s.* per yard; what will the whole come to?

Ans. 49*l.* 17*s.* 0¼*d.*

OF

VAULTED AND ARCHED ROOFS.

ARCHED *roofs* are either *vaults, domes, saloons*, or *groins*.

Vaulted roofs are formed by arches springing from the opposite walls, and meeting in a line at the top.

Domes are made by arches springing from a circular or polygonal base, and meeting in a point at the top.

Saloons are formed by arches connecting the side walls to a flat roof, or ceiling, in the middle.

Groins are formed by the intersection of vaults with each other.

Domes and saloons rarely occur in the practice of measuring, but vaults and groins over the cellars of most houses.

Vaulted roofs are generally one of the three following sorts:

1. *Circular roofs*, or those whose arch is some part of the circumference of a circle.
2. *Elliptical roofs*, or those whose arch is some part of the circumference of an ellipsis.
3. *Gothic roofs*, or those which are formed by two circular arcs that meet in a point directly over the middle of the breadth, or span of the arch.

PROBLEM I.

To find the solid content of a circular, elliptic, or gothic vaulted roof.

RULE.*

Multiply the area of one end by the length of the roof, and the product will be the solidity required.

EXAMPLES.

1. What is the solid content of a semi-circular vault, whose span is 40 feet, and its length 120 feet?

.7854
1600 = *square of* 40.

4712400
7854

2)1256.6400

628.32 = *area of the end.*
120 = *length.*

75398.40 = *solidity required.*

2. Required the solidity of an elliptic vault, whose span is 40 feet, height 12 feet, and length 80?
Ans. 30159.36 *feet.*

3. What is the solid content of a gothic vault, whose span is 48, the chord of its arch 48, the distance of the arch from the middle of the chord 18, and the length of the vault 60?
Ans. 136224.71712.†

PROBLEM II.

To find the concave, or convex surface, of circular, elliptic, or gothic vaulted roofs.

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[*] *To find the solidity of the materials in either of the arches.*

Rule. From the solid content of the whole arch take the solid content of the void space, and the remainder will be the solidity of the arch.

[†] The areas of these segments were calculated by rule 2, Prob. XIII. and the triangle by Prob. VIII.

RULE.*

Multiply the length of the arch by the length of the vault, and the product will be the superficies required.

EXAMPLE.

What is the concave surface of a semi-circular vault, whose span is 40 feet and its length 120?

$$\begin{array}{r l}
3.1416 & \\
40 & \\
\hline
2)125.6640 & \\
\hline
62.832 & = \textit{length of the arch.} \\
120 & \\
\hline
7539.840 & = \textit{concave surface required.} \\
\hline
\end{array}$$

PROBLEM III.

To find the solid content of a dome; its height, and the dimensions of its base being known.

RULE.†

Multiply the area of the base by two-thirds of the height, and the product will be the solidity.

EXAMPLES.

1. What is the solid content of a spherical dome, the diameter of whose circular base is 60 feet?

---

[*] The convex surface of a vault may be found by stretching a string over it; but for the concave surface this method is not applicable, and therefore its length must be found from proper dimensions.

[†] Domes and saloons are of various figures, but they are things that seldom occur in the practice of measuring.

.7854
3600= *square of* 60.

4712400
23562

2827.4400= *area of the base.*
20= $\frac{2}{3}$ *of the height* (30).

56548.8000= *solidity required.*

2. In an hexagonal spherical dome, one side of the base is 20 feet: what is the solidity? Ans. 12000 *feet.*

PROBLEM IV.

To find the superficial content of a spherical dome.

RULE.*

Multiply the area of the base by 2, and the product will be the superficial content required.

EXAMPLES.

1. What will the painting of an hexagonal spherical dome come to, at 1*s.* per yard; each side of the base being 20 feet?

2.598076= *area of a hexagon whose side is* 1.
400= *square of* 20.

1039.230400= *area of the base.*
2

9)2078.460800= *superficial content required.*

2.0)230.940088

11.5470044= 11*l.* 10*s.* 11*d.* *the expense of painting.*

---

* *The practical rule for elliptical domes is as follows:*

RULE. Add the height to half the diameter of the base, and this sum multiplied by 1.5708 will give the superficial content *nearly.*

PROBLEM V.

To find the solid content of a saloon.

RULE.*

1. Multiply the height of the arc, its projection, one-fourth of the perimeter of the ceiling, and 3.1416 continually together, and call the product A.

2. From a side or diameter of the room take a like side or diameter of the ceiling, and multiply the square of the remainder by the proper factor, (page 63,) and this product again by two-thirds of the height, and call the last product B.

3. Multiply the area of the flat ceiling by the height of the arch, and this product added to the sum of A and B will give the content required.

EXAMPLES.

1. What is the solid content of a saloon with a circular quadrantal arch of 2 feet radius, springing over a rectangular room of 20 feet long, and 16 feet wide?

---

* *To find the superficial content of a saloon.*

1. Find the area of the flat part of the ceiling.

2. Find the convex surface of a cylinder, or cylindroid, whose length is equal to one-fourth the perimeter of the ceiling, and its diameters to twice the height and twice the projection of the arch.

3. Find the superficial content of a dome of the same figure as the arch, and whose base is either a square, or a figure similar to that of the ceiling; the side being equal to the difference of a side of the room and a side of the ceiling.

4. Add these three articles together, and the sum will give the superficial content required.

Note.—In a rectangular, circular, or polygonal room, the base or the dome will be a square, a circle, or a like polygon.

Here the flat part of the ceiling is 16 *feet by* 12; *and*

4)56

14=¼ *of the perimeter.*
2=*height.*

28
2=*projection.*

56
3.1416
56

188496
157080

175.9296=A.
20=*side of the room.*
16=*side of the ceiling.*

4
4

16
1.000, &c.=*factor.*

16.000
1⅓=⅔ *of the height.*

16.000
5.333

21.333=B.
16
12

192=*area of the flat ceiling.*
2=*height of the arch.*

384

```
384
175.9296
 21.3333
--------
581.2629= solid content required.
```

2. A circular building of 40 feet diameter, and 25 feet high to the ceiling, is covered with a saloon, whose circular quadrantal arch is 5 feet radius; required the capacity of the room in cubic feet. Ans. 30779.45948 *feet.*

PROBLEM VI.

To find the solid content of the vacuity formed by a groin arch, either circular or elliptical.

RULE.

Multiply the area of the base by the height, and the product again by .904, and it will give the solidity required.

EXAMPLES.

1. What is the solid content of the vacuity formed by a circular groin, one side of its square base being 12 feet?

Here $12^2 \times 6 \times .904 = 781.056 =$ *solidity required.*

2. What is the solid content of the vacuity formed by an elliptical groin, one side of its square base being 20 feet, and the height 6 feet? Ans. 2169.6.

PROBLEM VII.

To find the concave superficies of a circular groin.

RULE.*

Multiply the area of the base by 1.1416, and the product will be the superficies required.

---

* This rule may also be observed in elliptical groins, the error being too small to be regarded in practice.

In measuring works where there are many groins in a range the cylindrical pieces between the groins, and on their sides, must be computed separately.

And to find the solidity of the brick or stone work, which forms the groin arches, observe the following

RULE. Multiply the area of the base by the height, including the work over the top of the groin, and this product lessened by the solid content, found as before, will give the solidity required.

The general rule for measuring all arches, is this:

From the content of the whole, considered as solid, from the springing of the arch to the outside of it, deduct the vacuity contained between the said springing and the under side of it, and the remainder will be the content of the solid part.

And because the upper sides of all arches, whether vaults or groins, are built up solid, above the haunches, to the same height with the crown, it is evident that the area of the base will be the whole content above mentioned, taking for its thickness the height from the springing to the top. And for the content of the vacuity to be deducted, take the area of its base, accounting its thickness to be two-thirds of the greatest inside height. But it may be noted that the area used in the vacuity, is not exactly the same with that used in the solid; for the diameter of the former is twice the thickness of the arch less than that of the latter.

And although I have mentioned the deduction of the vacuity as common to both the vault and the groin, it is reasonable to make it only in the former, on account of the waste of materials and trouble to the workman, in cutting and fitting them for the angles and intersections.

Whoever wishes to see this subject more fully handled, may consult *La Theorie et la Pratique de la Geometrie, par M. l'Abbe Deidier;* a work in which several parts of Mensuration and Practical Geometry are skilfully handled, the examples being mostly wrought out in an easy familiar manner, and illustrated with observations, and figures very neatly executed.

EXAMPLES.

1. What is the curve superficies of a circular groin arch, one side of its square being 12 feet?

Here $12^2 \times 1.1416 = 164.3904 =$ *superficies required.*

2. What is the concave superficies of a circular groin arch, one side of its square being 9 feet? Ans. 92.4696.

OF THE

CARPENTER'S RULE.

THIS instrument is commonly called *Cogeshall's* sliding rule. It consists of two pieces, of a foot in length each, which are connected together by means of a folding joint.

On one side of the rule, the whole length is divided into inches and half quarters, for the purpose of taking dimensions. And on this face there are also several plain scales, divided by diagonal lines into twelve parts, which are designed for planning such dimensions as are taken in feet and inches.

On one part of the other face there is a slider, and four lines marked A, B, C, and D; the two middle ones B and C being upon the slider.

Three of these lines A, B, C, are double ones, because they proceed from 1 to 10 twice over: and the fourth line D is a single one, proceeding from 4 to 40, and is called the girt *line*.

The use of the double lines A, and B, is for working proportions, and finding the areas of plane figures. And the use of the girt line D, and the other double line C, is for measuring solids.

When 1 at the beginning of any line is counted 1, then the 1 in the middle will be 10, and the 10 at the end 100. And when 1 at the beginning is counted 10, then the 1 in the middle is 100, and the 10 at the end 1000, &c. and all the small divisions are altered in value accordingly.

Upon the other part of this face, there is a table of the value of a load of timber, at all prices, from 6d. to 2s. a foot.

Some rules have likewise a line of inches, or a foot divided decimally into 10th parts; as well as tables of board measure, &c. but these will be best understood from a sight of the instrument.

THE USE OF THE SLIDING RULE.

PROBLEM I.

To find the product of two numbers, as 7 *and* 26.

RULE.

Set 1 upon A, to one of the numbers (26) upon B; then against the other number (7) on A, will be found the product (182) upon B.

Note.—If the third term runs beyond the end of the line, seek it on the other radius, or part of the line, and increase the product 10 times.

PROBLEM II.

To divide one number by another, as 510 *by* 12.

RULE.

Set the divisor (12) on A, to 1 on B; then against the dividend (510) on A, is the quotient (42½) on B.

Note.—If the dividend runs beyond the end of the line, diminish it 10 or 100 times to make it fall on A, and increase the quotient accordingly.

PROBLEM III.

To square any number, as 27.

RULE.

Set 1 upon D to 1 upon C; then against the number (27) upon D, will be found the square (729) upon C.

If you would square 270, reckon the 1 on D to be 100; and then the 1 on C will be 1000, and the product 72900.

PROBLEM IV.

To extract the square root of any number, as 4268.

RULE.

Set 1 upon C, to 1 upon D; then against (4268) the number on C, is (65.3) the root on D.

To value this right you must suppose the 1 on C to be some of these squares, 1, 100, 1000, &c. which is the nearest to the given number, and then the root corresponding will be the value of the 1 upon D.

PROBLEM V.

To find a mean proportional between any two numbers, as 27 *and* 450.

RULE.

Set one of the numbers (27) on C, to the same on D; then against the other number (450) on C, will be the mean (110.2) on D.

Note.—If one of the numbers overruns the line, take the 100th part of it, and augment the answer 10 times.

PROBLEM VI.

Three numbers being given, to find a fourth proportional; suppose 12, 28, *and* 57.

RULE.*

Set the first number (12) upon A, to the second (28) upon B; then against the third number (57) on A, is the fourth (133) on B.

Note.—If one of the middle numbers runs off the line, take the 10th part of it only, and augment the answer 10 times.

The finding a third proportional is exactly the same, the second number being twice repeated.

Thus, suppose a third proportional was required to 21 and 32.

Set the first 21 on B, to the second 32 on A; then against the second 32 on B, is 48.8 on A, which is the third proportional required.

---

* The use of the rule in board and timber measure will be shown in what follows.

If the breadth of a board be given; to find how much in length will make a square foot.

Rule. If the board be narrow, it will be found in the table of board measure on the rule; but, if not, shut the rule, and seek the breadth in the line of board measure, running along the rule, from that table; then over against it, on the opposite side, is the length in inches required.

The side of the square of a piece of timber being given; to find how much in length will make a foot solid.

Rule. If the timber be small, it will be found in the table of timber measure on the rule; but, if not, look for the side of the square, in the line of timber measure, running along the rule, from that table, and against it in the line of inches is the length required.

OF

TIMBER MEASURE.

PROBLEM I.

To find the area, or superficial content of a board or plank.

RULE.

MULTIPLY the length by the breadth, and the product will be the content required.

Note.—When the board is tapering, add the breadths of the two ends together, and take half the sum for the mean breadth.

BY THE SLIDING RULE.

Set 12 on B to the breadth in inches on A, then against the length in feet on B is the content on A, in feet and fractional parts as required.

EXAMPLES.

1. What is the value of a plank, whose length is 8 feet 6 inches, and breadth 1 foot 3 inches throughout, at 2½d. per foot ?

feet	in.		
8	6		
1	3		
8	6		
2	1	6	
10	7	6	the content.

	10	7	6
2d. is ⅙	1	8	
½ is ¼		5	
in. 6 is ½		1¼	
in. 1 is ⅙		¼	

2s. 2½d. the Answer.

BY THE SLIDING RULE.

As 12 on B : 15 on A : : 8½ on B : 10½ on A.

2. What is the content of a board, whose length is 5 feet 7 inches, and breadth 1 foot 10 inches?

feet. in. pa.
Ans 10 2 10.

3. At 1½d. per foot, what is the value of a plank whose length is 12 feet 6 inches, and breadth 11 inches throughout? Ans. 1s. 5d.

4. Find the value of 5 oaken planks at 3d. per foot, each being 17½ feet long, and their particular breadths as follows: viz. two of 13½ inches in the middle, one of 14½ inches in the middle, and the two remaining ones, each 18 inches at the broader end, and 11¼ at the narrower?

Ans. 1l. 5s. 9¼d.

PROBLEM II.

To find the solidity of squared or four-sided timber.

RULE.*

Multiply the mean breadth by the mean thickness, and this product again by the length, and it will give the solidity required.

---

* *Note* 1. If the stick be equally broad and thick throughout, the breadth and thickness, any where taken, will be the mean breadth and thickness.

BY THE SLIDING RULE.

As the length in feet on C : 22 on D : : quarter girt in inches on D : solidity on C.

EXAMPLES.

1. The length of a piece of timber is 20½ feet, the breadth at the greater end is 1 foot 9 inches, and the thickness 1 foot 3 inches; and at the less end the breadth is 1 foot 6 inches, and the thickness 1 foot: what is the solidity?

1.75= *greater breadth.*
1.5= *less breadth.*

2)3.25

1.625= *mean breadth.*

---

2. If the tree tapers regularly from one end to the other, the breadth and thickness, taken in the middle, will be the mean breadth and thickness.

3. If the stick does not taper regularly, but is thicker in some places than in others, let several different dimensions be taken, and their sum divided by the number of them will give the mean dimensions.

This method of finding the mean dimensions is mostly used in practice, but, in many cases, it is exceedingly erroneous.

The quarter girt, likewise, which is mentioned in the proportion by the sliding rule, is subject to error. It is not the fourth part of the circumference, but the square root of the product arising from multiplying the mean breadth by the mean thickness.

In order to show the fallacy of taking one-fourth of the girt for the side of a mean square, take the following example:

Suppose a piece of timber to be 24 feet long, and a foot square throughout, and let it be slit into two equal parts, from end to end.

Then the sum of the solidities of the two parts, by the quarter girt method, will be 27 feet, but the true solidity is 24 feet; and if the two pieces were very unequal, the difference would be still greater.

1.25= *greater thickness.*
1.00= *less thickness.*

2)2.25

1.125= *mean thickness.*

Now 1.625 × 1.125 × 20.5= 37.4765625= *content required.*

By Cross Multiplication.

fe.	*in.*	*pa.*		
1	7	6		
1	1	6		
1	7	6		
	1	7	6	
		9	9	
1	9	11	3	
20	6			
36	6	9	0	
	10	11	7	6
37	5	8	7	6= *content.*

BY THE SLIDING RULE.

As 1 *upon* B : $19\frac{6}{12}$ *upon* A : : $13\frac{6}{12}$ *upon* B : $263\frac{25}{100}$ *upon* A, *the mean square.*

As 16 *upon* C : 4 *upon* D : : 1.8 *upon* C : 16.2 *upon* D, *the side of the mean square.*

As $20\frac{1}{2}$ *upon* C : 12 *upon* D : : 16.2 *upon* D : $37\frac{5}{12}$ *upon* C, *the answer.*

2. The length of a piece of timber is 24.5 feet, and its ends are equal squares, whose sides are each 1.04 feet : what is the solidity ? Ans. 26 feet 6 inches.

3. The length of a piece of timber is 20.38 feet, and the ends are unequal squares, the side of the greater being

19⅛ inches, and that of the less 9⅞ inches: what is the solidity?

Ans. 29.756 *feet.*

4. The length of a piece of timber is 27.36 feet; at the greater end the breadth is 1.78 feet, and the thickness 1.23 feet; and at the less end the breadth is 1.04 feet and the thickness .91 feet: what is the solidity?

Ans. 41.278 feet.

PROBLEM III.

To find the solidity of round or unsquared timber.

RULE I.*

Multiply the square of the quarter girt (or one-fourth of the circumference) by the length, and the product will be the content, *according to the common practice.*

---

* Let c=girt or circumference, and l=length of the tree.

Then $\frac{c}{4} \times \frac{c}{4} \times l = \frac{c^2 l}{16}$=content of the tree according to the rule.

And $\frac{c^2}{4 \times 3.1416} \times l = \frac{c^2 l}{12.5664}$=true content, according to the rule for finding the content of a cylinder.

But $\frac{c^2 l}{12.5664}$ differs from $\frac{c^2 l}{16}$ by nearly one-fourth part of the whole, and therefore the rule is exceedingly erroneous.

When the tree is tapering, the mean girt is found in the same manner as in board measure. Or if the tree be very irregular, the best way is to divide it into a certain number of lengths, and find the content of each part separately.

When trees have their bark on, an allowance is generally made, by deducting so much from the girt as is judged sufficient to reduce it to such a circumference as it would have without its bark. In oak this allowance is about $\frac{1}{10}$ or $\frac{1}{12}$ part of the girt; but for elm, beach, ash, &c. whose bark is not so thick, the deduction ought to be less.

BY THE SLIDING RULE.

As the length upon C : 12 *upon* D : : ¼ *girt upon* D : *content upon* C.

EXAMPLES.

1. A piece of timber is 9¾ feet long, and the quarter girt is 39 inches; what is the solidity?

By Decimals.

Here 39 in. = 3.25 ft., *and* 9¾ ft. = 9.75 ft.

Hence $\overline{3.25}|^2 \times 9.75 = 10.5625 \times 9.75 = 102.984375$ *cubic feet* = *solidity required.*

By Cross Multiplication.

ft.	in.			
3	3			= 39 inches.
3	3			
9	9			
	9	9		
10	6	9		
9	9			= 9¾ feet.
95	0	9		
7	11	0	9	
102	11	9	9	= solidity.

BY THE SLIDING RULE.

As 9¾ *upon* C : 12 *upon* D : : 39 *upon* D : 103 *upon* C, *the content.*

2. The length of a tree is 25 feet, and the girt throughout 2½ feet; what is its solidity? Ans. 9 feet 9 inches.

3. The length of a tree is 14½ feet, and its girt in the middle 3.15 feet; required the solidity?

Ans. 9 feet nearly.

4. The girts of a tree in 4 different places are as follows: in the first place 5 feet 9 inches, in the second 4 feet 6 inches, in the third 4 feet 9 inches, and in the fourth 3 feet 9 inches; and the length of the whole tree is 15 feet; what is the solidity? Ans. 20 feet 7 inches.

5. An oak tree is 45 feet 7 inches long, and its quarter girt 3 feet 8 inches; what is the solid content, allowing $\frac{1}{12}$ for the bark? Ans. 515 feet, nearly.

RULE II.*

Multiply the square of one-fifth of the girt by twice the length, and the product will be the solidity, *extremely near the truth.*

---

* Let c=circumference, and 1=length, as before.

Then $\frac{c}{5} \times \frac{c}{5} \times 2l = \frac{2c^2 l}{25} = \frac{c^2 l}{12.5}$ = content of the tree according to the rule.

And the true content is $= \frac{c^2 l}{12.5664}$, as was before shown.

But $\frac{c^2 l}{12.5}$ differs from $\frac{c^2 l}{12.5664}$ by only about $\frac{1}{190}$ part of the whole, and is therefore sufficiently near the truth for any practical purpose.

This rule is full as easy in practice as the false one, and therefore ought to be generally used, since the ease of the other method is the only argument which is alleged for employing it.

The following rule was given me by *Mr. Burrow*, and is a still nearer approximation.

Rule. Multiply the square of the circumference by the length, and take $\frac{1}{11}$ of the product: from this last number subtract $\frac{1}{8}$ of itself, and the remainder will be the answer.

For $\frac{c^2 l}{12.5664} = \frac{7}{88}$ of $c^2 l$ *very nearly*, $= (\frac{8}{88} - \frac{1}{88}) \times c^2 l = \frac{c^2 l}{11} - \frac{1}{8}$ of $\frac{c^2 l}{11}$, which is the same as the rule, and differs from the truth by only 1 foot in 2300.

BY THE SLIDING RULE.

As twice the length upon C : 12 *upon* D : : *one-fifth of the girt upon* D : *content upon* C.

EXAMPLES.

1. A piece of timber is $9\frac{3}{4}$ feet long, and $\frac{1}{5}$ of the girt is 2.6 feet; what is the solidity?

By Decimals.

```
    2.6
    2.6
   ----
    156
    52
   ----
   6.76
   9.75
   ----
   3380
  4732
 6084
 ------
65.9100
      2
 ------
```

Here $\overline{2.6}|^2 \times 9.75 \times 2 = 131.8200 =$ *content.*

BY THE SLIDING RULE.

As 19.15 *upon* C : 12 *upon* D : : $31\frac{1}{5}$ *in. upon* D : 132 *the content upon* C.

2. If the length of a tree be 24 feet, and the girt throughout 8 feet; what is the content?

Ans. 123 feet, nearly.

3. If a tree girt 14 feet at the thicker end, and 2 feet at the smaller end; required the solidity when the length is 24 feet? Ans. 123 feet, nearly.

4. A tree girts in five different places as follows: in the first place 9.43 feet, in the second 7.92 feet, in the third 6.15 feet, in the fourth 4.74 feet, and in the fifth 3.16 feet; and the whole length is 17¼ feet; what is the solidity?
Ans. 54.4249 feet.

OF SPECIFIC GRAVITY.

The specific gravities of bodies are their relative weights contained under the same given magnitude, as a cubic foot, a cubic inch, &c.

The specific gravities of several sorts of bodies are expressed by the numbers annexed to their names in the following table :

A table of the specific gravities of bodies.

Fine gold	19640	Brick	2000
Standard gold	18888	Light earth	1984
Quicksilver	14000	Solid gunpowder	1745
Lead	11325	Sand	1520
Fine silver	11091	Pitch	1150
Standard silver	10535	Dry box wood	1030
Copper	9000	Sea water	1030
Gun metal	8784	Common water	1000
Cast brass	8000	Dry oak	925
Steel	7850	Gunpowder, shaken	922
Iron	7645	Dry ash	800
Cast iron	7425	Dry maple	755
Tin	7320	Dry elm	600
Marble	2700	Dry fir	550
Common stone	2520	Cork	240
Loam	2160	Air	1¼

Note.—As a cubic foot of water weighs just 1000 ounces Avoirdupois, the numbers in this table express not only the specific gravities of the several bodies, but also the weight of a cubic foot of each, in Avoirdupois ounces; and hence, by proportion, the weight of any other quantity, or the quantity of any other weight, may be readily known.

PROBLEM I.

To find the magnitude of a body from its weight being given.

RULE.

As the tabular specific gravity of the body, is to its weight in Avoirdupois ounces,

So is one cubic foot, or 1728 cubic inches, to its content in feet, or inches, respectively.

EXAMPLES.

1. Required the content of an irregular block of common stone which weighs 1 cwt. or 112 lbs.

Here 112 lbs. = 1792 oz.

oz. oz. c. in. c. in.
Hence 2520 : 1792 : : 1728 : $1228\frac{4}{5}$. Ans.

2. How many cubic inches of gunpowder are there in one pound weight? Ans. 30 nearly.

3. How many cubic feet are there in a ton weight of dry oak? Ans. $38\frac{138}{185}$.

PROBLEM II.

To find the weight of a body from its magnitude being given.

RULE.

As one cubic foot, or 1728 cubic inches, is to the content of the body,

So is its tabular specific gravity, to the weight of the body.

EXAMPLES.

1. Required the weight of a block of marble, whose length is 63 feet, and its breadth and thickness each 12 feet; these being the dimensions of one of the stones in the walls of Balbec.

Here $12^3 \times 63 = 9072$ c. ft. = content of the body.

Hence $1 : 9072 :: 2700 : 244944$ oz. $= 638\frac{7}{16}$ *tons, which is equal to the burthen of an East India ship.*

2. What is the weight of a pint of gunpowder, ale measure? Ans. 19 oz. nearly.

3. What is the weight of a block of dry oak, which measures 10 feet in length, 3 feet in breadth, and $2\frac{1}{2}$ feet deep? Ans. $4335\frac{15}{16}$ lbs.

PROBLEM III.

To find the specific gravity of a body.

RULE.

Case 1. When the body is heavier than water, weigh it both in water and out of water, and the difference will be the weight lost in the water.

Then, as the weight lost in water, is to the whole weight,

So is the specific gravity of water, to the specific gravity of the body.

EXAMPLES.

A piece of stone weighed in air 10 pounds, but in water only $6\frac{3}{4}$ pounds. Required its specific gravity.

$$
\begin{array}{rcrcrc}
10 & & & & & \\
6\frac{3}{4} & & & & & \\
\hline
3\frac{1}{4} & : & 10 & :: & 1000 & : \\
13 & : & 40 & :: & 1000 & : \\
 & & & & 40 & \\
\hline
\end{array}
$$

$$
\begin{array}{l}
13)40000)3077 \\
\quad 39 \\
\quad \overline{100} \\
\quad\ \ 91 \\
\quad \overline{\ \ 90}
\end{array}
$$

Case 2. When the body is lighter than water, so that it will not quite sink, affix to it another body heavier than water, so that the mass compounded of the two may sink together.

Weigh the heavier body and the compound mass separately both in water and out of it, and find how much each loses in water, by subtracting its weight in water from its weight in air.

Then as the difference in these remainders is to the weight of the light body in air,

So is the specific gravity of water to the specific gravity of the body.

EXAMPLE.

Suppose a piece of elm weighs in air 15 pounds, and that a piece of copper which weighs 18 pounds in air, and 16 pounds in water, is affixed to it, and that the compound weighs 6 pounds in water; required the specific gravity of the elm.

18 in air	33
16 in water	6
loss 2	27 loss.
	2
	25

25 : 15 : : 1000 : 600 Ans.

PROBLEM IV.

To find the quantities of two ingredients in a given compound.

RULE.

Take the difference of every pair of the three specific gravities, viz. of the compound and each ingredient, and multiply the difference of every two by the third.

Then as the greatest product is to the whole weight of the compound, so is each of the other products to the weights of the two ingredients.

EXAMPLE.

A composition of 112 pounds being made of tin and copper, whose specific gravity is found to be 8784; required the quantity of each ingredient: the specific gravity of tin being 7320, and of copper 9000.

9000	9000	8784
7320	8784	7320
1680	216	1464 diff.
8784	7320	9000
702720	4320	13176000
52704	648	
8784	1512	
14757120	1581120	

14757120 : 112 :: 13176000

112

26352000

13176

13176

14757120)1475712000(100

Ans. 100lb. of copper and 12lb. of tin } in the composition.

MISCELLANEOUS QUESTIONS.

1. WHAT difference is there between a floor 48 feet long, and 30 feet broad, and two others each of half the dimensions? Ans. 720 feet.

2. From a mahogany plank 26 inches broad, a yard and a half is to be sawed off; what distance from the end must the line be struck? Ans. 6.23 feet.

3. A joist is 8½ inches deep, and 3½ broad; what will be the dimensions of a scantling just as big again as the joist, that is 4¾ inches broad? Ans. 12.52 inches deep.

4. A roof is 24 feet 8 inches by 14 feet 6 inches, and is to be covered with lead at 8lbs. to the foot; what will it come to at 18s. per cwt? Ans. 22*l.* 19s. 10¼d.

5. What is the side of that equilateral triangle, whose area cost as much paving at 8d. per foot, as the palisading the three sides did at a guinea per yard?
Ans. 72.746 feet.

6. The two sides of an obtuse-angled triangle are 20 and 40 poles; what must the length of the third side be that the triangle may contain just an acre?
Ans. 58.876, or 23.099.

7. If two sides of an obtuse-angled triangle, whose area is $60\sqrt{3}$, be 12 and 20; what is the third side?
Ans. 28.

8. If an area of 24 be cut off from a triangle, whose three sides are 13, 14, and 15, by a line parallel to the longest side; what are the lengths of the sides including that area? Ans. $\frac{13}{7}\sqrt{14}$, $2\sqrt{14}$, and $\frac{15}{7}\sqrt{14}$.

9. The distance of the centres of two circles, whose diameters are each 50, is equal to 30; what is the area of the space enclosed by their circumference?
Ans. 559.115.

10. There is a segment of a circle the chord of which is 60 feet, and versed sine 10 feet; what will be the versed sine of that segment of the same circle whose chord is 90 feet? Ans. 28.2055.

11. The area of an equilateral triangle, whose base falls on the diameter, and its vertex in the middle of the arc of a semicircle is equal to 100; what is the diameter of the semicircle? Ans. 26.32148.

12. The four sides of a field, whose diagonals are equal to each other, are 25, 35, 31, and 19 poles, respectively; what is the area?* Ans. 4 ac. 1 ro. 38 poles.

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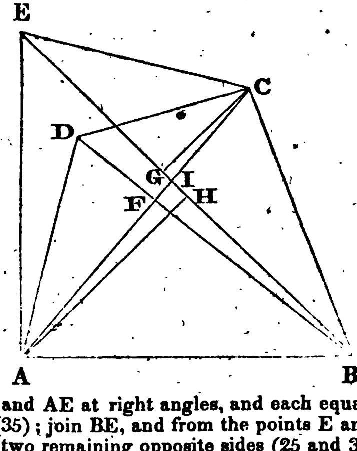
[*] *Construction.* In this question the sums of the squares of the opposite sides of the trapezium being equal, the figure may be constructed as follows:



Draw AB and AE at right angles, and each equal to one of the given sides (35); join BE, and from the points E and B, with radii equal to the two remaining opposite sides (25 and 31) respectively, describe arcs intersecting in C on the farther side of BE; join AC, and draw BF at right angles to it. With the centre C, and radius equal to the remaining side (19) describe an arc cutting BF produced in D. Join AD and CD, then will ABCD be the figure required.

Demonstration.—By the question $AB^2+CD^2=BC^2+CE^2$ and since BD and AC cross each other at right angles, (47.1) $AB^2+CD^2=BC^2+AD^2$; wherefore $AD^2=EC^2$ or $AD=EC$.

13. A cable which is 3 feet long, and 9 inches in compass, weighs 22 pounds, what will a fathom of that cable weigh whose diameter is 9 inches? Ans 434.25 lbs.

14. A circular fish-pond is to be dug in a garden that shall take up just half an acre, what must the length of the chord be that strikes the circle? Ans. 27.75 yards.

15. A carpenter is to put an oaken curb to a round well, at 8d. per foot square; the breadth of the curb is to be 7¼ inches, and the diameter within 3½ feet: what will be the expense? Ans 5s. 2¼d.

Hence, in the two triangles ABD and EAC, we have the two sides BA, AD equal the two AE, EC, each to each, and the angles ABD and EAC equal (each being the complement of BAF) and EC and AD, similarly situated; wherefore BD=AC.

Q. E. D.

Calculation.—On BE let fall the perpendiculars CG and AH; now $BE^2 = AB^2 + AE^2 = 35^2 \times 2$; $BE = \sqrt{35^2 \times 2} = 35\sqrt{2} = 49.4975$; and $AH = BH = \frac{1}{2}BE = 24.7487$; again (13.2) $BG = \frac{BC^2 + BE^2 - CE^2}{2BE} = \frac{31^2 + 2 \times 35^2 - 25^2}{2 \times 49.4975} = \frac{2786}{98.995} = 28.1428$; $GH = BG - BH = 28.1428 - 24.7487 = 3.3941$; $CG = \sqrt{BC^2 - BG^2} = \sqrt{31^2 - 28.1428^2} = \sqrt{168.98280816} = 12.9993$. By sim. triangles $AH + CG (37.748) : GH (3.3941) :: AH (24.7487) : HI = 2.2253$; $AI = \sqrt{AH^2 + HI^2} = \sqrt{24.7487^2 + 2.2253^2} = \sqrt{612.49815169 + 4.95196009} = \sqrt{617.45011178} = 24.8485$. Again, by sim. triangles, $HI (2.2253) : HG (3.3941) :: AI (24.8485) : AC = 37.8997 = BD$; now, by Problem V. Superficies, the area of the trapezium $ABCD = \frac{AC \times \overline{BF + FD}}{2} = \frac{AC \times BD}{2} = \frac{AC^2}{2} = \frac{37.8997^2}{2} = 718.1936$ po. $=$ 4 ac. 1 ro. 38 po. Ans.

Note.—This method is applicable to all questions of the kind wherein the diagonals cross each other at right angles; that is, when the sums of the squares of the opposite sides are equal, but a general solution to the question, without this, would involve an equation of the higher order.

16. Suppose the expense of paving a semicircular plot, at 2s. 4d. per foot, amounted to 10*l.* what is the diameter of it ? Ans. 14.7739.

17. Seven men bought a grinding stone of 60 inches in diameter, each paying $\frac{1}{7}$ part of the expense ; what part of the diameter must each grind down for his share ?*

Ans. { The 1st, 4.4508 ; 2d, 4.8400 ; 3d, 5.3535 ;
4th, 6.0765 ; 5th, 7.2079 ; 6th, 9.3935 ;
and the 7th, 22.6778.

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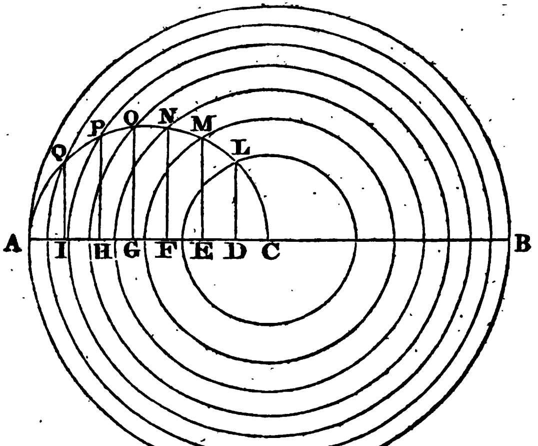
* This problem may be thus constructed:

On the radius AC describe a semicircle; also divide AC into as many equal parts CD, DE, EF, &c. as there are shares, and erect the perpendicular DL, EM, FN, &c. meeting the semicircle described on AC in L, M, N, O, P, Q. Then with the centre C and radii CL, CM, CN, &c. describe circles, and the thing is done.



For, by the nature of the circle, the square of the chords or radii CL, CM, CN, &c. are as the cosines CD, CE, CF, &c.

18. A gentleman has a garden 100 feet long, and 80 feet broad, and a gravel walk is to be made of an equal width half round it; what must the width of the walk be so as to take up just half the ground?*

Ans. 25.9688 feet.

19. In the midst of a meadow well stored with grass,
I took just an acre to tether my ass;
How long must the chord be, that feeding all round,
He may'nt graze less or more than an acre of ground?

Ans. 39.25073 yards.

20. A maltster has a kiln that is 16 feet 6 inches square; now he wants to pull it down, and build a new one that will dry three times as much at a time as the old one did; what must be the length of its side?

Ans. 28 feet 7 inches.

21. If a round cistern be 26.3 inches in diameter, and 52.5 inches deep; how many inches in diameter must a cistern be to hold twice the quantity, the depth being the same? Ans. 37.19 inches.

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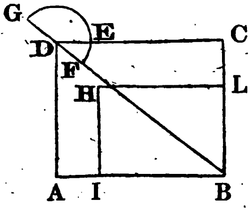
* This problem may be constructed thus: Let ABCD represent the garden, make CE=CB, and with the centre D and radius DE describe the semicircle GEF. Make BI=½ BG, BL=½ BF, and complete the rectangle IBLH, and the thing is done.



For the area of $\text{BIHL} = \text{BI.BL} = \frac{\text{BG}}{2} \times \frac{\text{BF}}{2} = \frac{\text{BD}+(\text{CD}-\text{CB})}{2}$

$$\frac{\text{BD}-(\text{CD}-\text{CB})}{2} = \frac{\text{BD}^2-(\text{CD}-\text{CB})^2}{4} = \frac{\text{BD}^2-(\text{CD}^2-2\text{CD.CB}+\text{CB}^2)}{4}$$

$= \frac{\text{BD}^2-\text{BD}^2+2\text{CD.CB}}{4} = \frac{\text{CD.CB}}{2} = \frac{1}{2}$ *the area of the garden.*

Q. E. D.

23. A may-pole, whose top was broken off by a blast of wind, struck the ground at 15 feet distance from the bottom of the pole: what was the height of the whole may-pole, supposing the length of the broken piece to be 39 feet? Ans. 75 feet.

24. What will the diameter of a globe be, when the solidity and superficial content thereof are represented by the same number? Ans. 6.

25. How many three inch cubes can be cut out of a 12 inch cube? Ans. 64.

26. A farmer borrowed part of a hay-rick of his neighbour, which measured 6 feet every way, and paid him back again by two equal cubical pieces, each of whose sides was three feet: Query, whether the lender was fully paid?
Ans. He was paid ¼ part only.

27. What will the painting of a conical church spire come to at 8d. per yard; supposing the circumference of the base to be 64 feet, and the altitude 118 feet?
Ans. 14*l.* 0s. 8¾d.

28. What will a marble frustum of a cone come to at 12s. per solid foot; the diameter of the greater end being 4 feet, that of the less end 1½ feet, and the length of the slant side 8 feet? Ans. 30*l.* 1s. 11¾d.

29. The diameter of a legal Winchester bushel is 18½ inches, and its depth 8 inches: what must the diameter of that bushel be whose depth is 7½ inches?
Ans. 19.1067.

30. Suppose the ball at the top of St. Paul's Church is 6 feet in diameter: what did the gilding of it come to at 3½d. per square inch? Ans. 237*l.* 10s. 1d.

31. A person wants a cylindric vessel of 3 feet deep, that shall hold twice as much as a vessel of 28 inches deep, and 46 inches in diameter: what must be the diameter of the vessel required? Ans. 57.37 inches.

32. Two porters agreed to drink off a quart of strong beer between them, at two pulls, or a draught each; now the first having given it a black eye, as it is called, or drank till the surface of the liquor touched the opposite edge of the bottom, gave the remaining part of it to the other: what was the difference of their shares, supposing the pot was the frustum of a cone, the depth being 5.7 inches, the diameter at the top 3.7 inches, and that of the bottom 4.23 inches? Ans. 7.05 cubic inches.

33. The monument erected in Babylon by Queen Semiramis at her husband Ninus's tomb, is said to have been one solid block of marble in the form of a pyramid; the base was a square whose side was 20 feet; and the height of the monument was 150 feet; now suppose this monument was sunk in the Euphrates, what weight would be sufficient to raise the apex of it to the surface of the water, and what weight would raise the whole of it above the water?

Ans. *To raise it to the surface would require* 948$\frac{37}{56}$ *tons, to raise it out would require* 1506$\frac{39}{56}$ *tons, (which is the same as the weight of the monument.)*

34. If the pyramid described in the last example were divided into three equal parts by planes parallel to its base, what would be the length of each part, beginning at the top?

Ans. 104.0042, 27.0329 and 18.9629 respectively.

35. How high above the surface of the earth must a person be raised to see $\frac{1}{3}$ of its surface?

Ans. *To the height of the earth's diameter.*

36. A cubical foot of brass is to be drawn into a wire of $\frac{1}{40}$ of an inch in diameter; what will be the length of the wire, allowing no loss in the metal?

Ans. 97784.5684 yards, or near 56 miles.

37. A gentleman has a bowling green, 300 feet long, and 200 feet broad, which he would raise one foot higher by means of the earth to be dug out of a ditch that goes round it; to what depth must the ditch be dug, supposing its breadth to be every where 8 feet?

Ans. 7$\frac{23}{86}$ feet.

38. Of what diameter must the bore of a cannon be, which is cast for a ball of 24lbs. weight, so that the diameter of the bore may be $\frac{1}{10}$ of an inch more than that of the ball? Ans. 5.647 inches.

39. One end of a certain pile of wood is perpendicular to the horizon, the other is in the form of an inclined plane: the length of the pile at the bottom is 64 feet, length at the top 50 feet, height 12 feet, length of the wood 5 feet; required the number of chords it contains? Ans. $26\frac{23}{32}$.

40. The ellipse in Grosvenor-square measures 840 links across the longest way, and 612 the shortest, within the walls: now the walls being 14 inches thick, it is required to find what ground they enclose, and what they stand upon.

Ans. *They enclose* 4ac. 0ro. 6po. *and stand on* 1760½ square feet.

41. If a heavy sphere whose diameter is 4 inches, be put into a conical glass, full of water, whose diameter is 5, and altitude 6 inches: it is required to know how much water will run over?

Ans. $\frac{35}{47}$ *of a pint nearly, or* 26.272 inches.

42. Suppose it be found by measurement, that a man of war, with its ordnance, rigging and appointments, draws so much water as to displace 50,000 cubic feet of water: required the weight of the vessel?

Ans. $1395\frac{5}{56}$ tons.

43. One ev'ning I chanc'd with a tinker to sit,
Whose tongue ran a great deal too fast for his wit:
He talked of his art with abundance of mettle;
So I ask'd him to make me a flat-bottom'd kettle.
Let the top and the bottom diameters be,
In just such proportion as five is to three:
Twelve inches the depth I propos'd, and no more;
And to hold in ale gallons seven less than a score.
He promis'd to do it, and straight to work went;
But when he had done it he found it too scant.

He alter'd it then, but too big he had made it;
For though it held right, the diameters fail'd it:
Thus making it often too big and too little,
The tinker at last had quite spoiled his kettle;
But declares he will bring his said promise to pass,
Or else that he'll spoil every ounce of his brass.
Now to keep him from ruin, I pray find him out
The diameter's length, for he'll ne'er do't, I doubt.

Ans. *The bottom diameter is* 14.4401, *and the top diameter* 24.4002.

44. If the above mentioned frustum of the cone were to hold as much again, what would be the length of the part added to the greater end?

Ans. 6.384 in. nearly.

A

TABLE

OF THE

AREAS OF THE SEGMENTS OF A CIRCLE,

Whose diameter is Unity, and supposed to be divided into 1000 equal Parts.

Versed Sine.	*Seg. Area.*	Versed Sine.	*Seg. Area.*	Versed Sine.	*Seg. Area.*
.001	.000042	.024	.004921	.047	.013392
.002	.000119	.025	.005230	.048	.013818
.003	.000219	.026	.005546	.049	.014247
.004	.000337	.027	.005867	.050	.014681
.005	.000470	.028	.006194	.051	.015119
.006	.000618	.029	.006527	.052	.015561
.007	.000779	.030	.006865	.053	.016007
.008	.000951	.031	.007209	.054	.016457
.009	.001135	.032	.007558	.055	.016911
.010	.001329	.033	.007913	.056	.017369
.011	.001533	.034	.008273	.057	.017831
.012	.001746	.035	.008638	.058	.018296
.013	.001968	.036	.009008	.059	.018766
.014	.002199	.037	.009383	.060	.019239
.015	.002438	.038	.009763	.061	.019716
.016	.002685	.039	.010148	.062	.020196
.017	.002940	.040	.010537	.063	.020680
.018	.003202	.041	.010931	.064	.021168
.019	.003471	.042	.011330	.065	.021659
.020	.003748	.043	.011734	.066	.022154
.021	.004031	.044	.012142	.067	.022652
.022	.004322	.045	.012554	.068	.023154
.023	.004618	.046	.012971	.069	.023659

The Areas of the Segments of a Circle.

Versed Sine.	Seg. Area.	Versed Sine.	Seg. Area.	Versed Sine.	Seg. Area.
.070	.024168	.102	.042080	.134	.062707
.071	.024680	.103	.042687	.135	.063389
.072	.025195	.104	.043296	.136	.064074
.073	.025714	.105	.043908	.137	.064760
.074	.026236	.106	.044522	.138	.065449
.075	.026761	.107	.045139	.139	.066140
.076	.027289	.108	.045759	.140	.066833
.077	.027821	.109	.046381	.141	.067528
.078	.028356	.110	.047005	.142	.068225
.079	.028894	.111	.047632	.143	.068924
.080	.029435	.112	.048262	.144	.069625
.081	.029979	.113	.048894	.145	.070328
.082	.030526	.114	.049528	.146	.071033
.083	.031076	.115	.050165	.147	.071741
.084	.031629	.116	.050804	.148	.072450
.085	.032186	.117	.051446	.149	.073161
.086	.032745	.118	.052090	.150	.073874
.087	.033307	.119	.052736	.151	.074589
.088	.033872	.120	.053385	.152	.075306
.089	.034441	.121	.054036	.153	.076026
.090	.035011	.122	.054689	.154	.076747
.091	.035585	.123	.055345	.155	.077469
.092	.036162	.124	.056003	.156	.078194
.093	.036741	.125	.056663	.157	.078921
.094	.037323	.126	.057326	.158	.079649
.095	.037909	.127	.057991	.159	.080380
.096	.038496	.128	.058658	.160	.081112
.097	.039087	.129	.059327	.161	.081846
.098	.039680	.130	.059999	.162	.082582
.099	.040276	.131	.060672	.163	.083320
.100	.040875	.132	.061348	.164	.084059
.101	.041476	.133	.062026	.165	.084801

The Areas of the Segments of a Circle.

Versed Sine.	*Seg. Area.*	Versed Sine.	*Seg. Area.*	Versed Sine.	*Seg. Area.*
.166	.085544	.198	.110226	.230	.136465
.167	.086289	.199	.111024	.231	.137307
.168	.087036	.200	.111823	.232	.138150
.169	.087785	.201	.112624	.233	.138995
.170	.088535	.202	.113426	.234	.139841
.171	.089287	.203	.114230	.235	.140688
.172	.090041	.204	.115035	.236	.141537
.173	.090797	.205	.115842	.237	.142387
.174	.091554	.206	.116650	.238	.143238
.175	.092313	.207	.117460	.239	.144091
.176	.093074	.208	.118271	.240	.144944
.177	.093836	.209	.119083	.241	.145799
.178	.094601	.210	.119897	.242	.146655
.179	.095366	.211	.120712	.243	.147512
.180	.096134	.212	.121529	.244	.148371
.181	.096903	.213	.122347	.245	.149230
.182	.097674	.214	.123167	.246	.150091
.183	.098447	.215	.123988	.247	.150953
.184	.099221	.216	.124810	.248	.151816
.185	.099997	.217	.125634	.249	.152680
.186	.100774	.218	.126459	.250	.153546
.187	.101553	.219	.127285	.251	.154412
.188	.102334	.220	.128113	.252	.155280
.189	.103116	.221	.128942	.253	.156149
.190	.103900	.222	.129773	.254	.157019
.191	.104685	.223	.130605	.255	.157890
.192	.105472	.224	.131438	.256	.158762
.193	.106261	.225	.132272	.257	.159636
.194	.107051	.226	.133108	.258	.160510
.195	.107842	.227	.133945	.259	.161386
.196	.108636	.228	.134784	.260	.162263
.197	.109430	.229	.135624	.261	.163140

The Areas of the Segments of a Circle.

Versed Sine.	*Seg. Area.*	Versed Sine.	*Seg. Area.*	Versed Sine.	*Seg. Area.*
.262	.164019	.294	.192684	.326	.222277
.263	.164899	.295	.193596	.327	.223215
.264	.165780	.296	.194509	.328	.224154
.265	.166663	.297	.195422	.329	.225093
.266	.167546	298	.196337	.330	.226033
.267	.168430	.299	.197252	.331	.226974
.268	.169315	.300	.198168	.332	.227915
.269	.170202	.301	.199085	.333	.228858
.270	.171089	.302	.200003	.334	.229801
.271	.171978	.303	.200922	.335	.230745
.272	.172867	.304	.201841	.336	.231689
.273	.173758	.305	. 02761	.337	.232634
.274	.174649	.306	. 03683	.338	.233580
.275	.175542	.307	.204605	.339	.234526
.276	.176435	308	.205527	.340	.235473
.277	.177330	.309	.206451	.341	.236421
.278	.178225	.310	.207376	.342	.237369
.279	.179122	.311	.208301	.343	.238318
.280	.180019	.312	.209227	.344	.239268
.281	.180918	.313	.210154	.345	.240218
.282	.181817	.314	.211082	.346	.241169
.283	.182718	.315	.212011	.347	.242121
.284	.183619	.316	.212940	.348	.243074
.285	.184521	.317	.213871	.349	.244026
.286	.185425	.318	.214802	.350	.244980
.287	.186329	.319	.215733	.351	.245934
.288	.187234	.320	.216666	.352	.246889
.289	.188140	.321	.217599	.353	.247845
.290	.189047	.322	.218533	.354	.248801
.291	.189955	.323	.219468	.355	.249757
.292	.190864	.324	.220404	.356	.250715
.293	.191775	.325	.221340	.357	.251673

The Areas of the Segments of a Circle.

Versed Sine.	*Seg. Area.*	Versed Sine.	*Seg. Area.*	Versed Sine.	*Seg. Area.*
.358	.252631	.390	.283592	.422	.315016
.359	.253590	.391	.284568	.423	.316004
.360	.254550	.392	.285544	.424	.316992
.361	.255510	.393	.286521	.425	.317981
.362	.256471	.394	.287498	.426	.318970
.363	.257433	.395	.288476	.427	.319959
.364	.258395	.396	.289453	.428	.320948
.365	.259357	.397	.290432	.429	.321938
.366	.260320	.398	.291411	.430	.322928
.367	.261284	.399	.292390	.431	.323918
.368	.262248	.400	.293369	.432	.324909
.369	.263213	.401	.294349	.433	.325900
.370	.264178	.402	.295330	.434	.326892
.371	.265144	.403	.296311	.435	.327882
.372	.266111	.404	.297292	.436	.328874
.373	.267078	.405	.298273	.437	.329866
.374	.268045	.406	.299255	.438	.330858
.375	.269013	.407	.300238	.439	.331850
.376	.269982	.408	.301220	.440	.332843
.377	.270951	.409	.302203	.441	.333836
.378	.271920	.410	.303187	.442	.334829
.379	.272890	.411	.304171	.443	.335822
.380	.273861	.412	.305155	.444	.336816
.381	.274832	.413	.306140	.445	.337810
.382	.275803	.414	.307125	.446	.338804
.383	.276775	.415	.308110	.447	.339798
.384	.277748	.416	.309095	.448	.340793
.385	.278721	.417	.310081	.449	.341787
.386	.279694	.418	.311068	.450	.342782
.387	.280668	.419	.312054	.451	.343777
.388	.281642	.420	.313041	.452	.344772
.389	.282617	.421	.314029	.453	.345768

The Areas of the Segments of a Circle.

Versed Sine.	*Seg. Area.*	Versed Sine.	*Seg. Area.*	Versed Sine.	*Seg. Area.*
.454	.346764	.470	.362717	.486	.378701
.455	.347759	.471	.363715	.487	.379700
.456	.348755	.472	.364713	.488	.380700
.457	.349752	.473	.365712	.489	.381699
.458	.350748	.474	.366710	.490	.382699
.459	.351745	.475	.367709	.491	.383699
.460	.352741	.476	.368708	.492	.384699
.461	.353739	.477	.369707	.493	.385699
.462	.354736	.478	.370706	.494	.386699
.463	.355732	.479	.371705	.495	.387699
.464	.356730	.480	.372764	.496	.388699
.465	.357727	.481	.373703	.497	.389699
.466	.358725	.482	.374702	.498	.390699
.467	.359723	.483	.375702	.499	.391699
.468	.360721	.484	.376702	.500	.392699
.469	.361719	.485	.377701		

APPENDIX.

OF

GAUGING.

The business of cask-gauging is commonly performed by two instruments, namely, the gauging or sliding rule, and the gauging or diagonal rod.

1. OF THE GAUGING RULE.

This instrument serves to compute the contents of casks, &c. after the dimensions have been taken. It is a square rule, having various logarithmic lines on its four sides or faces, and three sliding pieces, running in grooves, in three of them.

Upon the first face are three lines, namely, two marked A, B, for multiplying and dividing; and the third, MD, for malt depth, because it serves to gauge malt. The middle one B is on the slider, and is a kind of double line, being marked at both the edges of the slider, for applying it to both the lines A and MD. These three lines are all of the same radius or distance from one to 10, each containing twice the length of the radius. A and B are placed and numbered exactly alike, each beginning at 1, which may be either 1, or 10, or 100, &c. or .1 or .01, or .001, &c. but whatever it is, the middle division, 10, will be ten times as much, and the last division 100 times as much. But 1 on the line MD is opposite 215, or more exactly 2150,4 on the other lines which number 2150.4 denotes the cubic inches in a malt bushel; and its divisions numbered retrograde to those of A and B. On these two lines are also several other marks and letters; thus, on the line A are MB, for malt bushel, at the number 2150.4; and A for ale, at 282, the cubic inches in an ale gallon; and on the line B is W, for wine, at 231, the cubic inches in a wine gallon; also, *si*, for square inscribed, at .707, the side of a square inscribed in a circle whose diameter is 1; *se*, for square equal, at .886, the side of a square which is equal to the same circle; and *c*, for circumference, at 3.1416, the circumference of the same circle.

On the second face, or that opposite the first, are a slider and four lines, marked D, C, D, E, at one end, and root, square, root, cube, at the other; the lines C and D containing respectively the square and cubes of the opposite numbers on the lines D, D; the radius of D being double to that of A, B, C, and triple to that of E; so that whatever the first 1 on D denotes, the first on C is the square of it, and the first on E the cube of it; so if D begin with 1, C and E will begin with 1; but if D begin with 10, C will begin with 100, and E with 1000; and so on. On the line C are marked *oc* at .0796, for the area of the circle, whose circumference is 1; and *od* at .7854, for the area of the circle whose diameter is 1. Also on the line D, are WG, for wine gauge, at 17.15; AG for ale gauge, at 18.95; and MR for malt round, at 52.32; these three being the gauge points for round and circular measure, and are found by dividing the square roots of 231, 282, and 2150.4 by the square root of .7854: also, MS, for malt square, are marked at 46.37, the malt gauge point for square measure, being the square root of 2150.4.

On the third face are three lines, one on a slider marked N; and two on the stock, marked SS and SL, for segment standing and segment lying, which serve for ullaging standing and lying casks.

And on the fourth, or opposite face, are a scale of inches, and three other scales marked spheroid, or 1st variety, 2d variety, 3d variety; the scale for the fourth or conic variety, being on the inside of the slider in the third face. The use of these lines is to find the mean diameters of casks.

Besides all those lines, there are two others on the insides of the first two sliders, being continued from the one slider to the other. The one of these is a scale of inches, from 12½ to 36; and the other is a scale of ale gallons, between the corresponding numbers 435 and 3.61; which form a table to show, in ale gallons, the contents of all cylinders whose diameters are from 12½ to 36 inches, their common altitude being 1 inch.

The use of the Gauging Rule.

PROBLEM I.

To multiply two numbers, as 12 *and* 25.

Set 1 on B, to either of the given numbers, as 12, on A; then against 25 on B, stands 300 on A; which is the product.

PROBLEM II.

To divide one number by another, as 300 *by* 25.

Set 1 on B, to 25 on A; then against 300 on A, stands 12 on B for the quotient.

PROBLEM III.

To find a fourth proportional, as to 8, 24, *and* 96.

Set A on B, to 24 on A; then against 96 on B, is 288 on A, the 4th proportional to 8, 24, 96 required.

PROBLEM IV.

To extract the square root, as of 225.

The first one on C standing opposite the one on D, on the stock; then against 225 on C, stands its square root 15 on D.

PROBLEM V.

To extract the cube root, as of 3375.

The line D on the slide being set straight with E; opposite 3375 on E stands its cube root 15 on D.

PROBLEM VI.

To find a mean proportional, as between 4 and 9.

Set 4 on C, to the same 4 on D; then against 9 on C, stands the mean proportional 6 on D.

PROBLEM VII.

To find numbers in duplicate proportion.

As, to find a number which shall be to 120 *as the square of* 3 *to the square of* 2.

Set 2 on D, to 120 on C; then against 3 on D, stands 270 on C; for the answer.

PROBLEM VIII.

To find numbers in subduplicate proportion.

As, to find a number which shall be to 2 *as the root of* 270 *to the root of* 120.

Set 2 on D, to 120 on C, then against 270 on C, stands 3 on D, for the answer.

PROBLEM IX.

To find the numbers in triplicate proportion.

As, to find a number which shall be to 100, *as the cube of* 36 *is to the cube of* 40.

Set 40 on D, to 100 on E; then against 36 on D, stands 72.9 on E, for the answer.

PROBLEM X.

To find numbers in subtriplicate proportion.

As, to find a number which shall be to 40, *as the cube root of* 72.9 *is to the cube root of* 100.

Set 40 on D, to 100 on E; then against 72.9 on E stands 36 on D, for the answer.

PROBLEM XI.

To compute malt bushels by the line MD.

As, to find the malt bushels in the couch, floor, or cistern, whose length is 230, *breadth* 58.2, *and depth* 5.4 *inches.*

Set 230 on B, to 5.4 on MD; then against 58.2 on A, stands 33.6 bushels on B, for the answer.

Note.—The uses of the other marks on the rule, will appear in the examples further on.

OF THE GAUGING OR DIAGONAL ROD.

The diagonal rod is a square rule, having four faces; being commonly four feet long, and folding together by joints. This instrument is used both for gauging or measuring casks, and computing their contents, and that from one dimension only, namely, the diagonal of the cask, or the length from the middle of the bung hole to the meeting of the head of the cask with the stave opposite to the bung; being the longest straight line that can be drawn within the cask from the middle of the bung. And, accordingly, on one face of the rule is a scale of inches for measuring this diagonal; to which are placed the areas, in ale gallons, of circles to the corresponding diameters, in like manner as the lines on the under sides of the three slides in the sliding rule.

On the opposite face, are two scales of ale and wine gallons, expressing the contents of casks having the corresponding diagonals. And these are the lines which chiefly form the difference between this instrument and the sliding rule; for all their other lines are the same, and are to be used in the same manner.

EXAMPLE.

The rod being applied within the cask at the bung hole, the diagonal was found to be 34.4 inches; required the content in gallons.

Now to 34.4 inches correspond, on the rod, 90¾ ale gallons or 111 wine gallons, the content required.

Note.—The contents exhibited by the rod, answer to the most common form of casks, and fall in between the 2d and 3d varieties following.

OF CASKS AS DIVIDED INTO VARIETIES.

It is usual to divide casks into four cases or varieties, which are judged of from the greater or less apparent curvature of their sides; namely,

1. The middle frustum of a spheroid,
2. The middle frustum of a parabolic spindle,
3. The two equal frustums of a paraboloid,
4. The two equal frustums of a cone.

And if the content of any of these be computed in inches, by their proper rules, and this be divided by 282, or 231, or 2150.4, the quotient will be the content in ale gallons or wine gallons, or malt bushels respectively. Because

282	cubic inches make	1 ale gallon
231	- - -	1 wine gallon
2150.4	- - -	1 malt bushel.

And the particular rule will be for each as in the following problems:

PROBLEM XII.

To find the content of a cask of the first form.

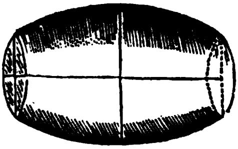
To the square of the head diameter add double the square of the bung diameter, and multiply the sum by the length of the cask. Then let the product

be multiplied by .0009¼, or divided by 1077, for ale gallons;

and multiplied by .0011⅓, or divided by 882 for wine gallons.

EXAMPLES.

1. Required the content of a spheroidal cask, whose length is 40, and bung and head diameters 32 and 24 inches.



24	32		
24	32		
96	64		
48	96		
576	1024		
	2		
	2048	104960	104960
	576	.0009¼	.0011⅓
	2624	944640	1154560
	40	26240	34987
	104960	ale 97.0880 gallons	118.9547 wine.

By the Gauging Rule.

Having set 40 on C, to the ale gauge 32.82 on D.
against
24 on D, stands 21.3 on C
32 on D, stands 38.0 on C
the same 38.0

sum 97.3 ale gallons.

And having set 40 on C, to the wine gauge 29.7 on D.
against
24 on D, stands 26.1 on C
32 on D, stands 46.5 on C
the same 46.5

119.1 wine gallons.

Ex. 2. Required the content of the spheroidal cask, whose length is 20, and diameters 12 and 16 inches.

Ans. { 12.136 ale gallons,
14.869 wine gallons.

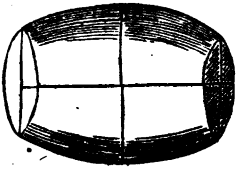
PROBLEM XIII.

To find the content of a cask of the second form.

To the square of the head diameter, add double the square of the bung diameter, and from the sum take $\frac{2}{5}$ or $\frac{4}{10}$ of the square of the difference of the diameters; then multiply the remainder by the length, and the product again by .0009¼ for ale gallons, or by .0011⅓ for wine gallons.

EXAMPLES.

1. The length being 40, and diameters 24 and 32, required the content.



32			
24			
8	2624.0	103936	103936
8	25.6	.0009¼	.0011⅓
64	2598.4	935424	1143296
4	40	25984	34645
25.6	103936	ale 96.1408 gall.	117.9741 wine.

By the Gauging Rule.

Having set 40 on C, to 32.82 on D, against 8 on D, stands 2.4 on C; the $\frac{4}{10}$ of which is 0.96. This taken from the 97.3 in the last form, leaves 96.3 ale gallons.

And having set 40 on C, to 29.7 on D, against 8 on D, stands 2.9 on C; the $\frac{4}{10}$ of which is 1.16. This taken from the 119.1 in the last form, leaves 117.9 wine gallons.

Ex. 2. Required the content when the length is 20, and the diameters 12 and 16.

Ans. { 12.018 *ale gallons.*
14.724 *wine gallons.*

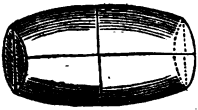
PROBLEM XIV.

To find the content of a cask of the third form.

To the square of the bung diameter add the square of the head diameter; multiply the sum by the length, and the product again by .0014 for ale gallons, or by .0017 for wine gallons.

EXAMPLES.

1. Required the content of a cask of the third form, when the length is 40, and the diameters 24 and 32.



1024	64000	64000
576	.0014	.0017
1600	256	448
40	64	64
64000	*ale* 89.6	108.8 *wine.*

By the Gauging Rule.

Set 40 on C, to 26.8 on D; then against
24 on D, stands 32.0 on C
32 on D, stands 57.3 on C

sum 89.3 *ale gallons.*

And having set 40 on C, to 24.25 on D; then
against 24 on D, stands 39.1 on C
32 on D, stands 69.8 on C

sum 108.9 wine gallons.

Ex. 2. Required the content when the length is 20, and the diameters 12 and 16.

Ans. { 11.2 ale gallons,
13.6 wine gallons.

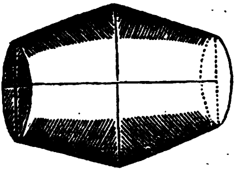
PROBLEM XV.

To find the content of a cask of the fourth form.

Add the square of the difference of the diameters to three times the square of their sum; then multiply the sum by the length, and the product again by .00023⅕ for ale gallons, or by .00028⅓ for wine gallons.

EXAMPLES.

1. Required the content, when the length is 40, and the diameters 24 and 32 inches.



56	8		
56	8		
		378880	378880
336	64	.00023⅕	.00028⅓
280	9408		
		1136640	3031040
3136	9472	757760	757760
3	40	75776	126293
9408	378880 ale	87.90016 gall.	107.34933 wine.

BY THE SLIDING RULE.

Set 40 on C, to 65.64 on D; then against
8 on D, stands 0.6 on C
56 on D, stands 29.1 on C
29.1
29.1

sum 87.9 ale gallons.

And set 40 on C, to 59.41 on D ; then against
8 on D, stands 0.7
56 on D, stands 35.6
35.6
35.6

sum 107.5 wine gall.

Ex. 2. What is the content of a conical cask, the length being 20, and the bung and head diameters 16 and 12 inches ?

Ans. { 10.985 ale gallons,
13.416 wine gallons.

PROBLEM XVI.

To find the content of a cask by four dimensions.

Add together the squares of the bung and head diameters, and the square of double the diameter taken in the middle between the bung and head : then multiply the sum by the length of the cask, and the product again by .0004⅔ for ale gallons, or by .0005⅔ for wine gallons.

EXAMPLES.

1. Required the content of any cask whose length is 40, the bung diameter being 32, the head diameter 24, and the middle diameter between the bung and the head 28¾ inches.

57.5	24	32
57.5	24	32
2875	96	64
4025	48	96
2875	576	1024
3306.25		
1024		
576		
4906.25		

4906.25
40

196250	196250
.0004$\frac{2}{3}$	.0005$\frac{2}{3}$
785000	981250
130833	130833

ale 91.5833 gallons 111.2083 wine.

BY THE SLIDING RULE.

Set 40 on C, to 46.4 on D; then agaiust
24 on D, stands 10.5
32 on D, stands 19.0
57½ on D, stands 62.0

sum 91.5 ale gallons.

Set 40 on C, to 24.0 on D; then against
24 on D, stands 13.0
32 on D, stands 23.2
57½ on D, stands 75.0

sum 111.2 wine gallons.

Ex. 2. What is the content of a cask, whose length is 20, the bung diameter being 16, the head diameter 12, and the diameter in the middle between them 14$\frac{3}{8}$?

Ans. { 11.4479 ale gallons,
13.9010 wine gallons.

PROBLEM XVII.

To find the content of any cask from three dimensions only.

Add into one sum 39 times the square of the bung diameter, 25 times the square of the head diameter, and 26 times the product of the two diameters: then multiply the sum by the length, and the product again by $\frac{00034.}{9}$ for wine gallons, or by $\frac{.00034}{11}$ or .00003$\frac{1}{11}$, for ale gallons.

EXAMPLES.

1. Required the content of a cask, whose length is 40, and the bung and head diameters 32 and 24.

```
       32        24        32
       32        24        24
       --        --      ----
       64        96       128
      96        48        64
     ----      ----      ----
     1024       576       768
       39        25        26
     ----      ----      ----
     9216      2880      4608
    3072      1152      1536
    -----     -----     -----
    39936     14400     19668
    -----     39936     -----
              19968
              -----
              74304
                 40
            -------
            2972160
             .00034          2972160
           --------          .00003 1/11
           11888640          ---------
           8916480            8916480
         ----------            270196
     9)1010.53440             --------
        112.2816 wine gall.   91.86676 ale gall.
```

Ex. 2. What is the content of a cask, whose length is 20, and the bung and head diameters 16 and 12?

Ans. { 11.4833 ale gallons,
{ 14.0352 wine gallons.

Note.—This is the most exact rule of any, for three dimensions only; and agrees nearly with the diagonal rod.